Analysis 2

23 April 2024

Warm-up: Give the derivative of

$$\sqrt{x^2 + 6x + 12}.$$

Derivative formulas

Last time

$f(x)$	$f'(x)$
x^p	$p\,x^{p-1}$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
e^x	(later)
$\ln(x)$	(later)

Constant Multiple: $(cf)' = cf'$

Sum Rule: $(f+g)' = f' + g'$

Product Rule:

$$(fg)' = fg' + f'g$$

Chain Rule:

$$\big(f(g)\big)' = f'(g) \cdot g'$$

From the Power Rule and algebra, $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\sqrt{x}\right] = \dfrac{1}{2\sqrt{x}}$.

The Chain Rule then tells us that $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\sqrt{\text{stuff}}\right] = \dfrac{1}{2\sqrt{\text{stuff}}} \cdot \text{stuff}'$.

Warmup:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\sqrt{x^2 + 6x + 12}\right] = \frac{1}{2\sqrt{x^2 + 6x + 12}} \cdot (2x + 6)$$

$$\left(\sqrt{x^2 + 6x + 12}\right)' = \frac{x + 3}{\sqrt{x^2 + x + 8}}$$

Finding local extremes

Last time

To find the local min/max of $f(x)$,

1. Find the CPs of f.

2. Compute signs of f' somewhere in between each critical point, and at one point with $x <$ all critical points, and at one point with $x >$ all CPs.

3. **The First Derivative Test**

- If $f' > 0$ just to the left of $x = c$ and $f' < 0$ just to the right of $x = c$, then f has a local maximum at $x = c$.
- If $f' < 0$ just to the left of $x = c$ and $f' > 0$ just to the right of $x = c$, then f has a local minimum at $x = c$.
- If f' has the same sign on both sides of $x = c$, then f has *neither* a local minimum nor local maximum at $x = c$.

Task 1: Find and classify* the critical point(s) of

$$f(x) = \sqrt{x^2 + 6x + 12}.$$

x	−4	−3	−3
f	↘	local min	↗
f′	−1/2	0	1/2

* Determine whether it is a local minimum, local maximum, or neither.

Task 2: Given that $x = \frac{3}{2}$ is a critical point of

$$f(x) = x^6 - \frac{9}{5}x^5 - \frac{15}{2}x^4 + 15x^3 + 18x^2 - 54x + 5,$$

classify it as a local minimum, local maximum, or neither.

$f'(1) < 0$ and $f'(2) > 0$, so you might think $x = 1.5$ is a min.

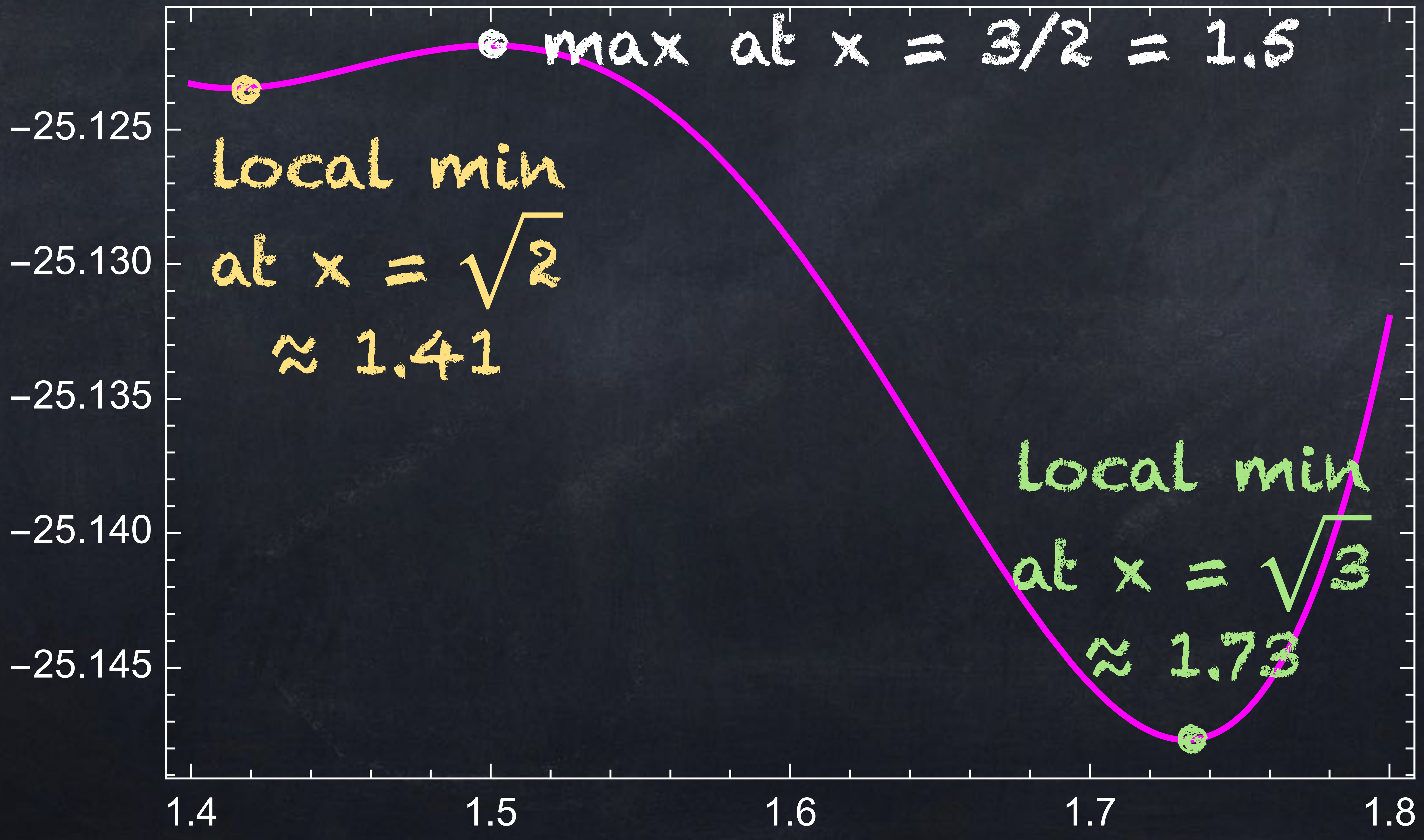
But in order to use the First Derivative Test we need to look at x-values in between critical points. In fact, $f(x)$ has another critical point at $\sqrt{2} \approx 1.414$, so the 1st D.T. would require us to look at an x-value in between $\sqrt{2}$ and 1.50.

$f'(1.42) > 0$ and $f'(1.6) < 0$ $\rightarrow$ $x = 1.5$ is a local max.

Task 2: Given that $x = \frac{3}{2}$ is a critical point of

$$f(x) = x^6 - \frac{9}{5}x^5 - \frac{15}{2}x^4 + 15x^3 + 18x^2 - 54x + 5,$$

classify it as a local minimum, local maximum, or neither.



Finding local extremes

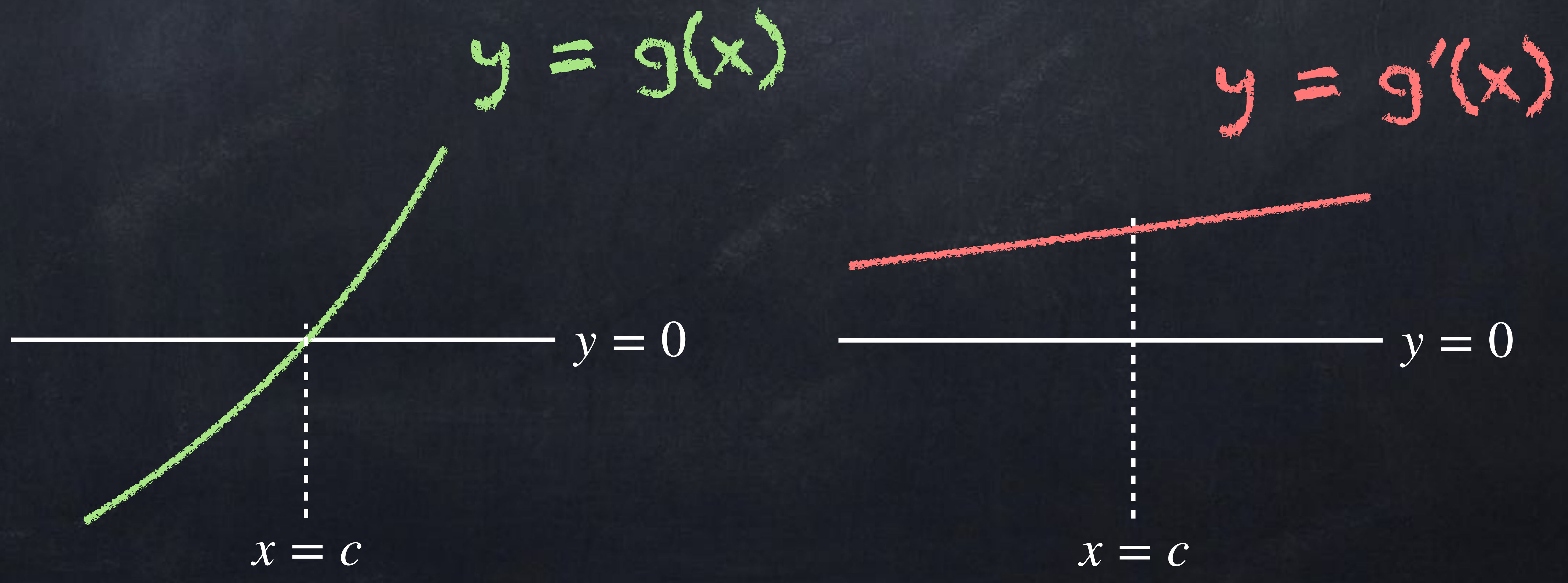
There are some problems with using the First Derivative Test to learn what kind of critical point $x = c$ is.

- Practical: we need to know *all* the CP of f in some interval in order to be sure we do not “skip over” any when calculating f' at points.

- Philosophical: we are looking for a *local* property, so why are we doing anything at points with $x \neq c$?

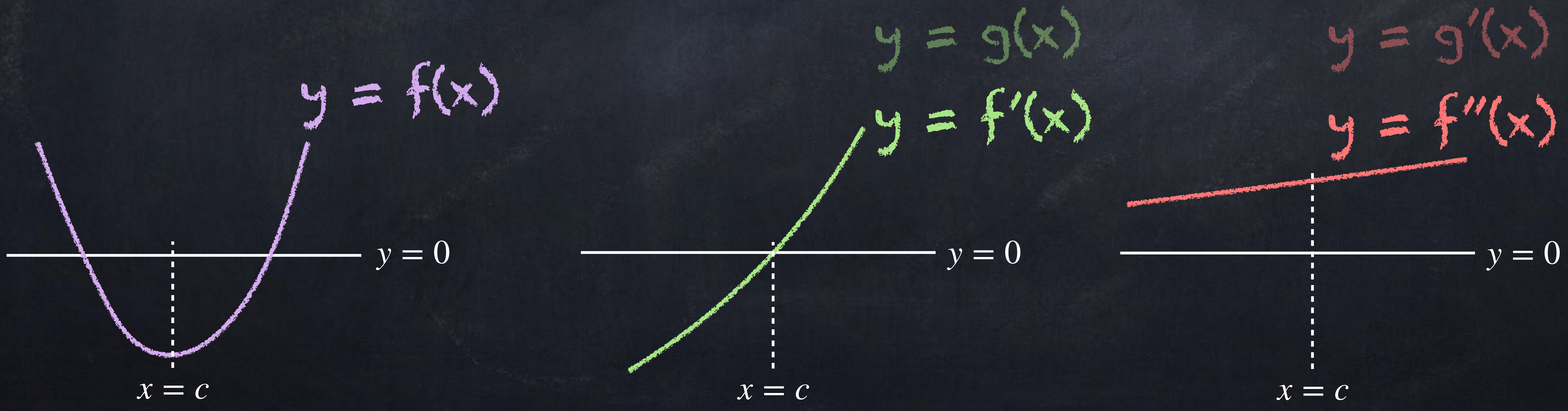
If $g(x)$ is negative for $x < 6$ and $g(x)$ is positive for $x > 6$ then…

- Could $g'(6)$ be positive?
- Could $g'(6)$ be negative?
- Could $g'(6)$ be zero?



Finding local extremes

In the graphs below, $f(x)$ has a local minimum:



Finding local extremes

To find the local min/max of $f(x)$,

1. Find the critical points of f.

2. Compute (the signs of) the values of f'' at each CP.

3. **The Second Derivative Test**

- If $f'(c) = 0$ and $f''(c) > 0$ then f has a local minimum at $x = c$. 
- If $f'(c) = 0$ and $f''(c) < 0$ then f has a local maximum at $x = c$. 

(The test does not help if $f''(c) = 0$.)

Finding local extremes

To find the local min/max of $f(x)$,

1. Find the critical points of f.

2. Compute signs of f' somewhere in each interval.
3. **The First Derivative Test**

or

2. Compute signs of f'' at each CP.
3. **The Second Derivative Test**

Task: Find the critical points of

$$f(x) = 2x^3 - \frac{3}{2}x^2 - 135x + 22$$

and classify each one as a local minimum or local maximum.

Answer: $x = -4.5$ is a local max

$x = 5$ is a local min

Concavity

These two functions are both increasing:

They both have $f'(x) > 0$, but clearly their behavior is different.

Concavity

There are several official definition for “concave up” and “concave down”.
We will just use pictures.

concave up

also called “convex” or “convex down”

down

concave down

also called “concave” or “convex up”

If $f''(x) > 0$ then f is concave up.

If $f''(x) < 0$ then f is concave down.

Image source: https://www.mathsisfun.com

Concavity

There are several official definition for “concave up” and “concave down”.
We will just use pictures.

concave up

also called “convex”
or “convex down”

concave down

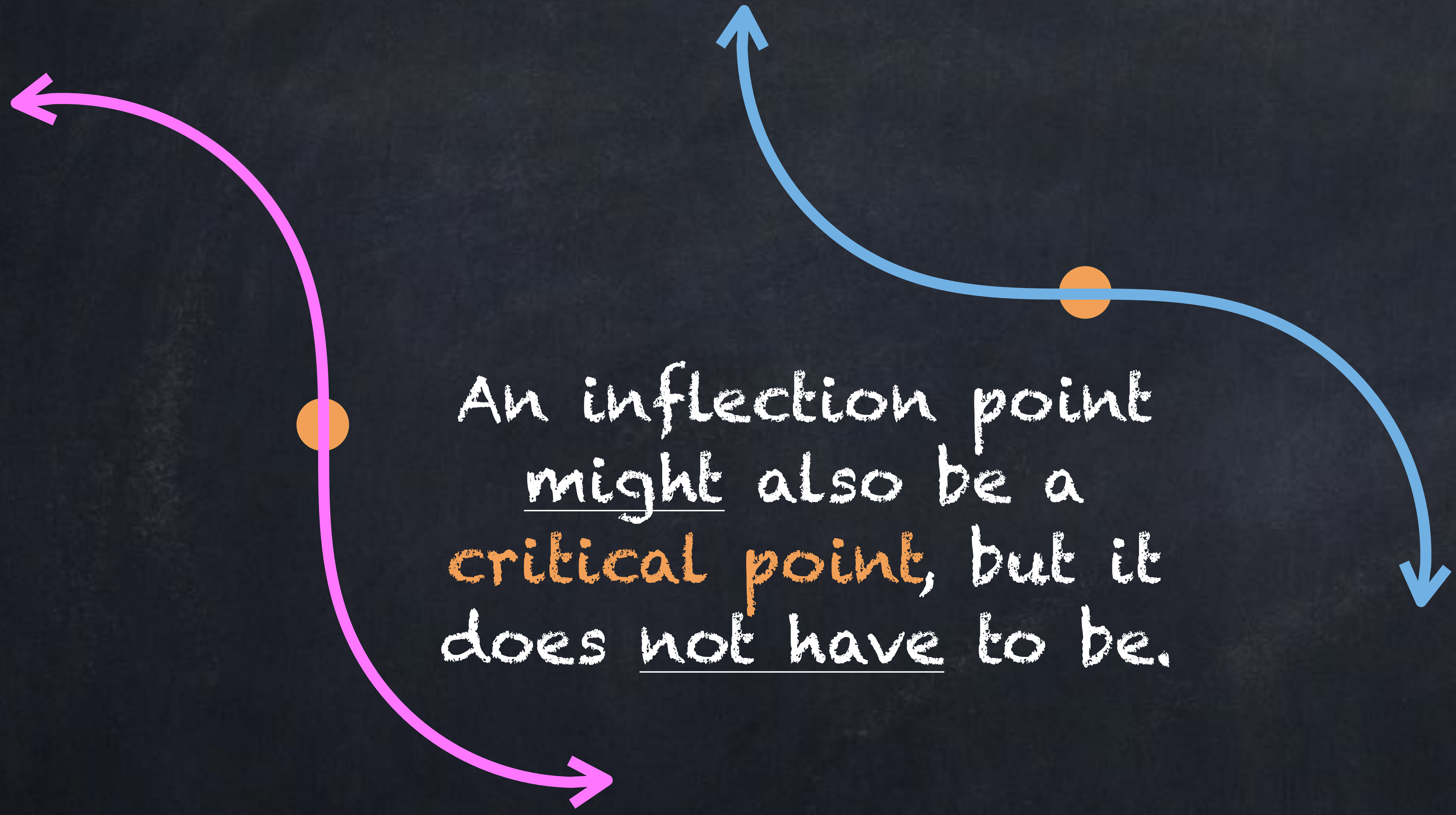
also called “concave”
or “convex up”

If $f''(x) > 0$ then f is concave up.

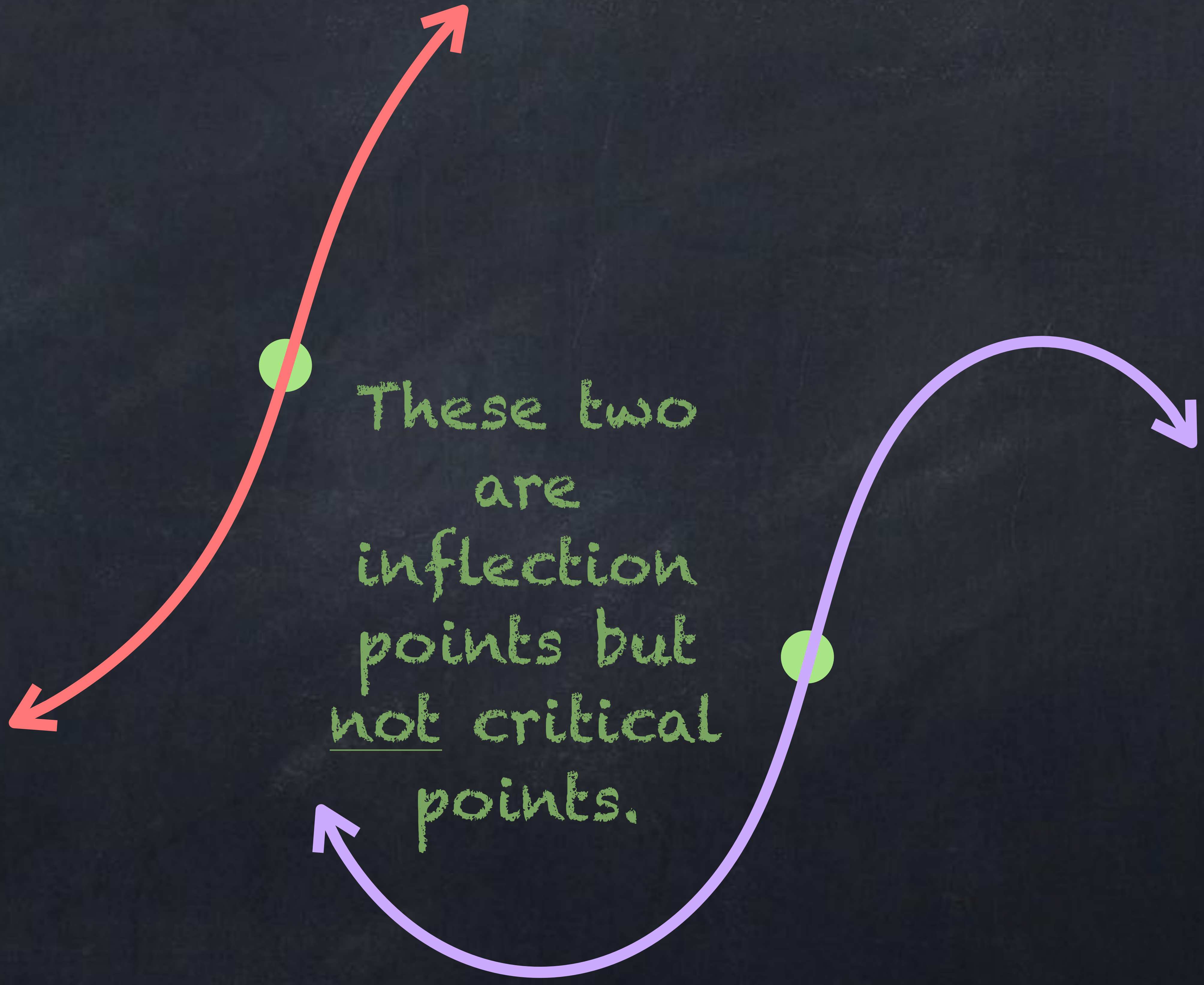
If $f''(x) < 0$ then f is concave down.

Concavity

Definition: an **inflection point** is a point where the concavity changes.



An inflection point might also be a critical point, but it does not have to be.



These two are inflection points but not critical points.

Seeing f′ and f″ in graphs

Last time

Monotonicity

- If $f' > 0$ then f is "increasing",
- If $f' < 0$ then f is "decreasing".
- An x-value* where f' *is zero* or *doesn't exist* is a "critical point".

Today

Concavity

- If $f'' > 0$ then f is "concave up",
- If $f'' < 0$ then f is "concave down".
- An x-value* where f'' *changes sign* is an "inflection point" (we will see examples next week).

* The x-value must to be in the domain of f.