Analysis 2

7 May 2024

Warm-up: If you run $114\,\mathrm{km}$ in $5\,\mathrm{hours}$, what is your "average speed"?

Speed (or velocity)

- If you run $114\,\mathrm{km}$ in 5 hours, what is your "average speed"?

$\frac{114}{5}$ = 22.8 kph (this is also 14.2 mph and 6.3 m/s)

- If your position in meters after t seconds is

$$y(t) = \frac{1}{10}t^2 + 2t,$$

calculate your "average speed" (in $\mathrm{m/s}$) between $t = 2$ and $t = 10$.

Speed (or velocity)

- If your position after t seconds is

$$y(t) = \frac{1}{10}t^2 + 2t,$$

<u>estimate</u> your "*instantaneous* speed" when $t = 2$.

$$\frac{y(2.1) - y(2)}{0.1} = \frac{4.641 - 4.4}{0.1} = 2.41\,\frac{m}{s}$$

$$\frac{y(2.001) - y(2)}{0.001} = 2.4001\,\frac{m}{s}$$

Speed (or velocity)

- If your position after t seconds is

$$y(t) = \frac{1}{10}t^2 + 2t,$$

calculate your "*instantaneous* speed" exactly when $t = 2$.

From the previous task, it seems like 2.4 might be the answer. But how can we know for sure?

- This requires a derivative or a limit.
- It really requires a limit because the official definition of derivative—which we have not seen yet—uses limits.

Analysis 1

Main topics:

- Limits
- Derivatives
  - rules for individual functions (power, trig)
  - rules for combining functions (Sum, Product, Quotient, Chain)
  - tangent lines
  - monotonicity (increasing vs. decreasing) and critical points
  - concavity (concave up vs. concave down) and inflection points
  - extrema (minima and maxima)
- Integrals

Analysis 1

Main topics:

- Limits
  - limit as $n \to \infty$ for a sequence
  - limit as $x \to \infty$ or $x \to -\infty$ for a function
  - limit as $x \to a$ for a function (a is some number)
  - limit as $x \to a^-$ or $x \to a^+$ for a function
  - graphs
  - calculations
- Derivatives
- Integrals

Weeks ago

For the function

$$f(x) = \frac{x^2 - x - 2}{x - 2},$$

when $x = 2$,

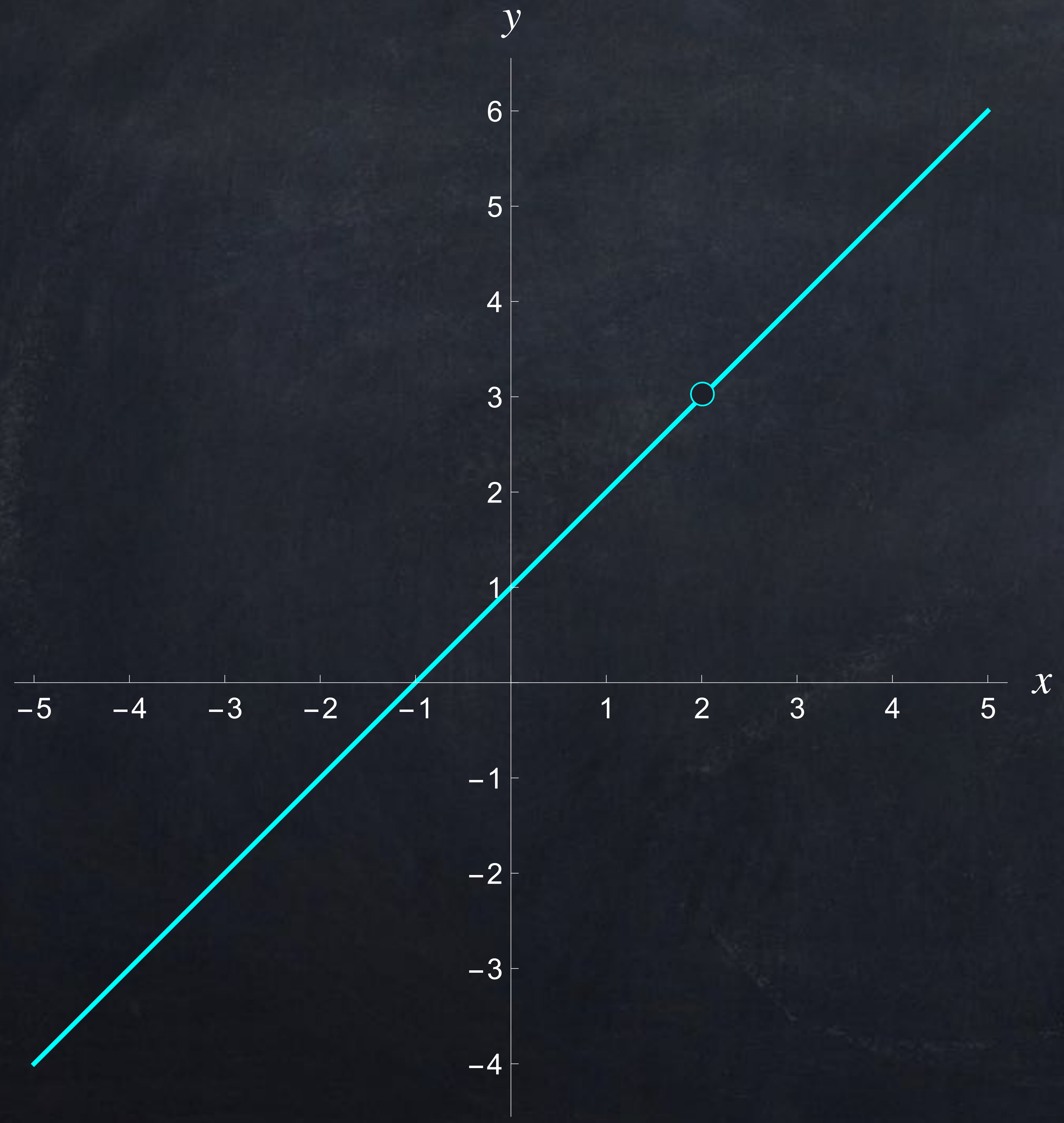
$$f(2) = \frac{2^2 - 2 - 2}{2 - 2} = \frac{0}{0} = 🙁.$$

But if we look at the graph $y = \dfrac{x^2 - x - 2}{x - 2}$, we will be able to say more about $f(2)$.

Weeks ago

For the function

$$f(x) = \frac{x^2 - x - 2}{x - 2},$$



All of the x-values very close to 2 give us values of $f(x)$ very close to 3.

In symbols, we write

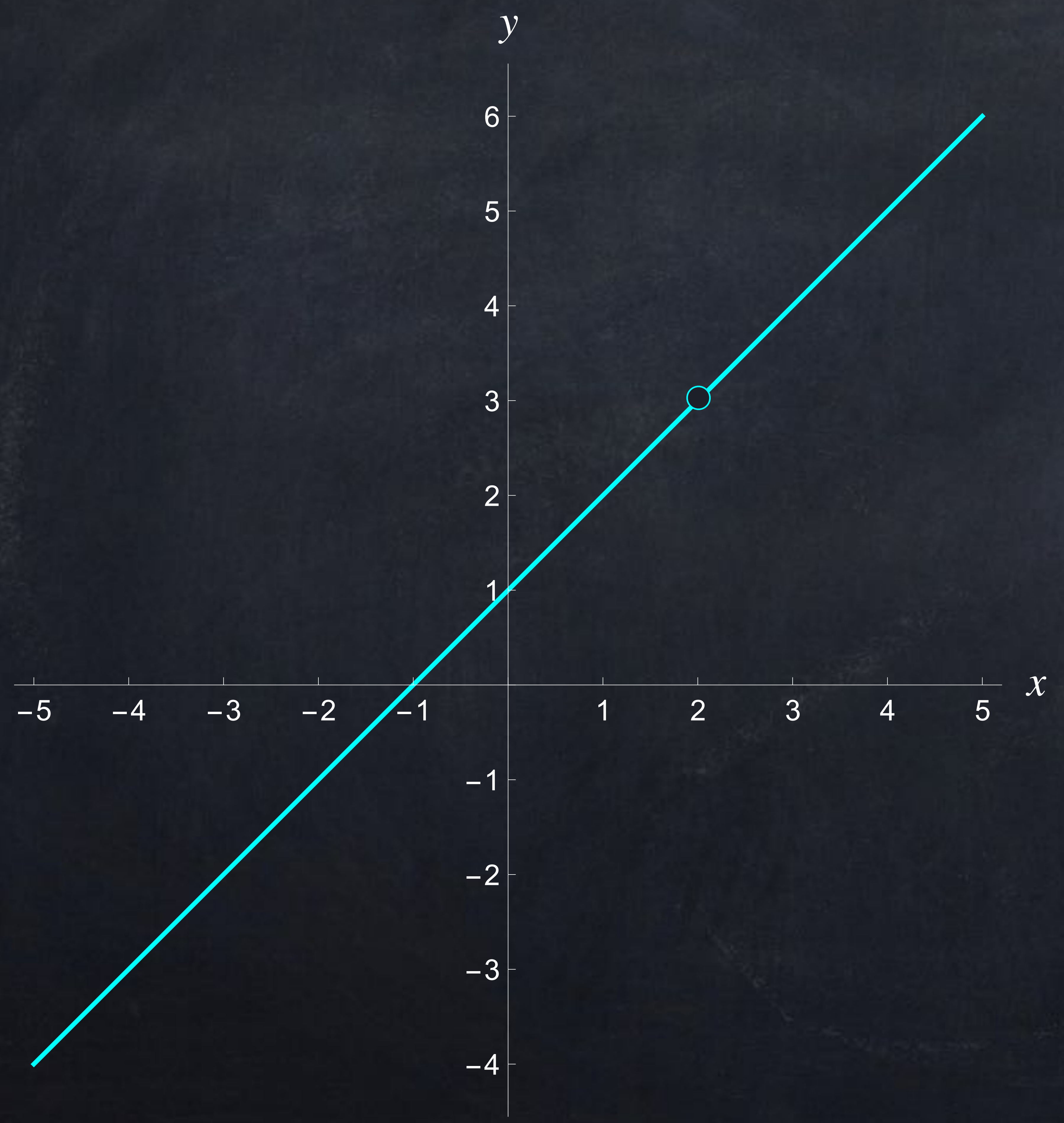
$$\lim_{x \to 2} f(x) = 3$$

for this function.

Weeks ago

For the function

$$f(x) = \frac{x^2 - x - 2}{x - 2},$$



All of the x-values very close to 2 give us values of $f(x)$ very close to 3.

In symbols, we write

$$\lim_{x \to 2} \frac{x^2 - x - 2}{x - 2} = 3.$$

For the function

$$f(x) = \frac{x^2 - x - 2}{x - 2},$$

we can also use a table of values to find $\lim_{x \to 2} f(x)$.

x	1.8	1.9	1.99	1.999	2.001	2.005	2.1
$f(x)$	2.8	2.9	2.99	2.999	3.001	3.005	3.1

These are very close to 3.

Note: this "limit" is about what happens when the input is CLOSE to a certain value but NOT exactly equal to it. We do NOT include x = 2 in this table.

Limits as x → a

In general, we write

$$\lim_{x \to a} f(x) = L,$$

if all values of x very close a give values of $f(x)$ that are very close to L.

The equation above is said out loud as

"the limit as X goes to A of F of X equals L"

or

"the limit as X approaches A of F of X equals L".

Limits as x → a

In general, we write

$$\lim_{x \to a} f(x) = L,$$

if all values of x very close a give values of $f(x)$ that are very close to L.

There is an official definition:

- $\lim_{x \to a} f(x) = L$ means that for any $w > 0$ there exists $d > 0$ such that if $a - d < x < a + d$ and $x \neq a$ then $L - w < f(x) < L + w$.

Limits as x → a

In general, we write

$$\lim_{x \to a} f(x) = L,$$

if all values of x very close a give values of $f(x)$ that are very close to L.

There is an official definition:

- $\lim_{x \to a} f(x) = L$ means that for any $\varepsilon > 0$ there exists $\delta > 0$ such that

  if $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$.

Often, this definition is written with absolute value notation…
and with Greek letters (epsilon ε and delta δ).

Looking at “instantaneous speed” earlier, we calculated

$$\frac{\left(\frac{1}{10}(2.1)^2+4\right)-\left(\frac{1}{10}(2)^2+4\right)}{2.1-2}=2.41$$

and

$$\frac{\left(\frac{1}{10}(2.001)^2+4\right)-\left(\frac{1}{10}(2)^2+4\right)}{2.001-2}=2.4001.$$

How do we know for sure that

$$\lim_{t\to 2}\frac{\left(\frac{1}{10}t^2+2t\right)-\left(\frac{1}{10}(2)^2+4\right)}{t-2}=2.4\ ?$$

How do we know for sure that $\displaystyle\lim_{x\to 2} \frac{\frac{1}{10}x^2 + 2x - 4.4}{x-2} = 2.4$?

- $\displaystyle\lim_{x\to a} f(x) = L$ means that for any $w > 0$ there exists $d > 0$ such that if $a-d < x < a+d$ and $x \neq a$ then $L-w < f(x) < L+w$.

$w = 0.1 \rightarrow$ If $\boxed{}$ then $2.3 < \dfrac{0.1x^2+2x-4.4}{x-2} < 2.5.$

$w = 0.03 \rightarrow$ If $\boxed{}$ then $2.37 < \dfrac{0.1x^2+2x-4.4}{x-2} < 2.43.$

$w = 0.00000512 \rightarrow$ If ... then $2.39999488 < \dfrac{0.1x^2+2x-4.4}{x-2} < 2.40000512.$

How do we know for sure that $\lim_{x \to 2} \dfrac{\frac{1}{10}x^2 + 2x - 4.4}{x - 2} = 2.4$?

- $\lim_{x \to a} f(x) = L$ means that for any $w > 0$ there exists $d > 0$ such that if $a-d < x < a+d$ and $x \neq a$ then $L-w < f(x) < L+w$.

$w = 0.1 \rightarrow$ If $1 < x < 3$ then $2.3 < \dfrac{0.1x^2+2x-4.4}{x-2} < 2.5.$

$w = 0.03 \rightarrow$ If $1.7 < x < 2.3$ then $2.37 < \dfrac{0.1x^2+2x-4.4}{x-2} < 2.43.$

$w = 0.00000512 \rightarrow$ If ... then $2.39999488 < \dfrac{0.1x^2+2x-4.4}{x-2} < 2.40000512.$

How do we know for sure that $\lim_{x \to 2} \dfrac{\frac{1}{10}x^2 + 2x - 4.4}{x - 2} = 2.4$?

- Use algebra.

$$\frac{0.1x^2 + 2x - 4.4}{x - 2}$$

$$= \frac{x^2 + 20x - 44}{10x - 20}$$

$$= \frac{(x - 2)(x + 22)}{10x - 20}$$

$$= \frac{x + 22}{10}$$

Then $\dfrac{(2) + 22}{10} = 2.4$ exactly

(no division by zero!).

A few important notes:

- $\lim\limits_{x \to 2} \dfrac{0.1x^2 + 2x - 4.4}{x - 2}$ *is* the number 2.4. It is not "approximately 2.4", and it is not something like 2.40000001.
- It doesn't matter whether there is any x-value for which $\dfrac{0.1x^2 + 2x - 4.4}{x - 2} = 2.4$ (in this example, there actually is no such x).

Example: find $\lim_{x \to 5} \dfrac{x-5}{x^2-25}$.

Method 1: table

x	4.9	4.95	4.99	4.999	5.001	5.005	5.02	5.1
$f(x)$								

Method 2: graph

Method 3: algebra

Limit properties

For any numbers a and c,

- $\lim_{x \to a} c = c$ and
- $\lim_{x \to a} x = a$.

Examples: $\lim_{x \to 6}(27) = 27$ and $\lim_{x \to 6}(x) = 6$.

These should not be surprising.

Limit properties

If the limits all exist and are finite, then

- $$\lim_{x\to a}\Big(f(x)+g(x)\Big) = \Big(\lim_{x\to a} f(x)\Big) + \Big(\lim_{x\to a} g(x)\Big),$$
- $$\lim_{x\to a}\Big(f(x)\cdot g(x)\Big) = \Big(\lim_{x\to a} f(x)\Big)\Big(\lim_{x\to a} g(x)\Big),$$
- $$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \quad \text{if } \lim_{x\to a} g(x) \neq 0,$$
- $$\lim_{x\to a} f\big(g(x)\big) = f\Big(\lim_{x\to a} g(x)\Big)$$ if f is a "nice" function.

with
$g(x) = c$

- $$\lim_{x\to a}\Big(c\cdot f(x)\Big) = c\cdot\Big(\lim_{x\to a} f(x)\Big)$$

Later we will see exactly when $\lim_{x\to a} f\big(g(x)\big) = f\Big(\lim_{x\to a} g(x)\Big)$ is allowed.

For now, it is enough to know that…

- any polynomial
    - This includes x^2.
- $\sqrt[n]{x}$
- $\sin(x)$ and $\cos(x)$
- e^x and a^x with $a > 0$
- $\ln(x)$ and $\log_b(x)$ with $b > 0$

$$\lim_{x\to a}\left(f(x)^2\right) = \left(\lim_{x\to a} f(x)\right)^2$$

can all be used safely in this limit rule.

*You might only be allowed to use $x \geq 0$ or $x > 0$ with these functions.

Example: Calculate $\lim_{x\to 3} x^2 - 15x + 9$ using the limit properties.

$$\lim_{x\to 3} x^2 - 15x + 9 = \left(\lim_{x\to 3} x^2\right) - \left(\lim_{x\to 3} 15x\right) + \left(\lim_{x\to 3} 9\right)$$

$$= \left(\lim_{x\to 3} x\right)^2 - 15\left(\lim_{x\to 3} x\right) + \left(\lim_{x\to 3} 9\right)$$

$$= (3)^2 - 15 \cdot (3) + (9)$$

$$= -27$$

This is same as the value of $x^2 - 15x + 9$ itself when $x = 3$.

I will say more later about when we can find limits just by plugging in an x value.

Limits as x → ±∞

The official definition

- $\lim_{x \to \infty} f(x) = L$ means that for any $\varepsilon > 0$ there exists an X such that if $x > X$ then $|f(x) - L| < \varepsilon$.

and a similar one for "$\lim_{x \to -\infty}$" can be difficult to understand at first, but these limits are very easy to see in graphs.