Analysis 2

14 May 2024

Warm-up: What is the derivative of $x^4 + 97.409$?

Warm-up: What is the derivative of $x^4 + \pi^4$?

Speed (or velocity)

Last week

- If your position after t seconds is

$$y(t) = \frac{1}{10}t^2 + 2t,$$

<u>estimate</u> your "*instantaneous* speed" when $t = 2$.

$$\frac{y(2.1) - y(2)}{0.1} = \frac{4.641 - 4.4}{0.1} = 2.41\ \frac{m}{s}$$

$$\frac{y(2.001) - y(2)}{0.001} = 2.4001\ \frac{m}{s}$$

Speed (or velocity)

Last week

- If your position after t seconds is

$$y(t) = \frac{1}{10}t^2 + 2t,$$

what is your "*instantaneous* speed" exactly when $t = 2$?

$$\lim_{t \to 2} \frac{\left(\frac{1}{10}t^2 + 2t\right) - \left(\frac{1}{10}(2)^2 + 2(2)\right)}{t - 2} = 2.4$$

We can find this value by first using algebra

to rewrite $\frac{(0.1t^2 + 2t) - (4.4)}{t - 2} = \frac{t + 22}{10}$.

Limits as x → a

Last week

In general, we write

$$\lim_{x \to a} f(x) = L,$$

to mean that

- *informally:* all values of x very close a give values of $f(x)$ that are very close to L.
- *official definition:* for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$.

We can often calculate these by using algebra first.

Example 1: Calculate $\lim_{x \to 3} \frac{(x+5)^2 - 25}{x}$. Answer: 13

Example 2: Calculate $\lim_{x \to 0} \frac{(x+5)^2 - 25}{x}$. Answer: 10

Example 3: Calculate $\lim_{x \to 3} \frac{\sqrt{x+13} - 4}{x-3}$. Answer: $\frac{1}{8}$

Note: we *cannot* say

$$\frac{0}{0} = 1 \quad \text{or} \quad \frac{0}{0} = 0$$

because, for example,

$$\lim_{x \to 0} \frac{(x+5)^2 - 25}{x} = 10 \quad \text{and} \quad \lim_{x \to 3} \frac{\sqrt{x+13} - 4}{x-3} = \frac{1}{8}$$

are both "$\frac{0}{0}$".

This is an example of an "indeterminant form". We will see more soon.

Limits as $x \to \pm\infty$

The official definition

- $\lim_{x\to\infty} f(x) = L$ means that for any $\varepsilon > 0$ there exists an X such that

  if $x > X$ then $|f(x) - L| < \varepsilon$.

and a similar one for " $\lim_{x\to-\infty}$ " can be difficult to understand at first, but these limits are very easy to see in graphs.

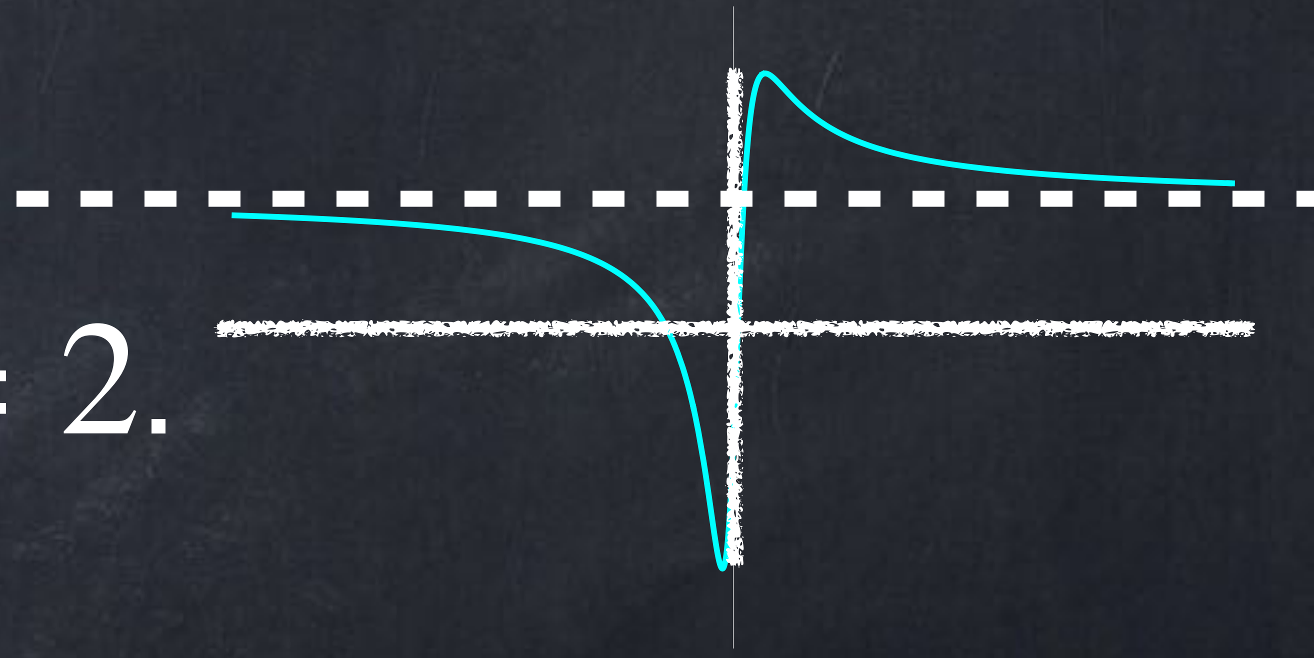
Limits as $x \to \pm\infty$

The line $y = c$ is a **horizontal asymptote** of the graph $y = f(x)$ if

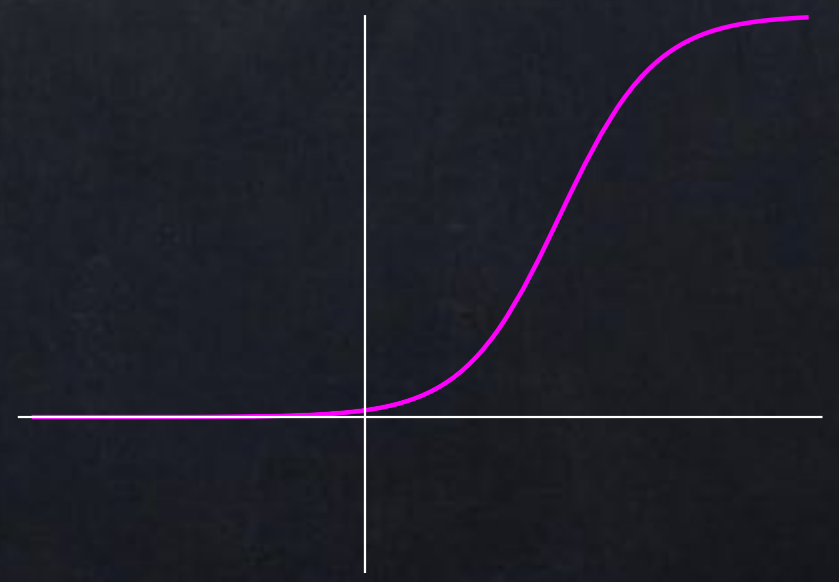
$$\lim_{x \to -\infty} f(x) = c \quad \text{or} \quad \lim_{x \to \infty} f(x) = c.$$

Examples:

- $f(x) = \dfrac{8x^2 + 30x - 9}{4x^2 + 5}$ has a horizontal asymptote at $y = 2$.



- $f(x) = \dfrac{10^x}{10^x + 58}$ has a horizontal asymptote at $y = 1$ and also at $y = 0$.



It is often helpful to think of

$$\infty - 5 = \infty, \qquad \frac{\infty}{2} = \infty, \qquad \frac{14}{\infty} = 0, \qquad \infty + \infty = \infty$$

sometimes, but be careful! We *cannot* say

$$\infty - \infty = 0 \qquad \text{or} \qquad \frac{\infty}{\infty} = 1$$

because, for example,

$$\lim_{x\to\infty} \frac{x+1}{2x} = \frac{1}{2}, \qquad \lim_{x\to\infty} \frac{2^x}{2x+1} = \infty, \qquad \lim_{x\to\infty} \frac{\sqrt{x}}{2x+1} = 0$$

are all "$\frac{\infty}{\infty}$".

There is no way to simplify $\frac{\infty}{\infty}$ that always works.

This is an example of an **indeterminate form**. Other indeterminate forms include

$$\infty - \infty, \qquad \frac{0}{0}, \qquad 0 \times \infty, \qquad 0^0, \qquad 1^\infty, \qquad \infty^0.$$

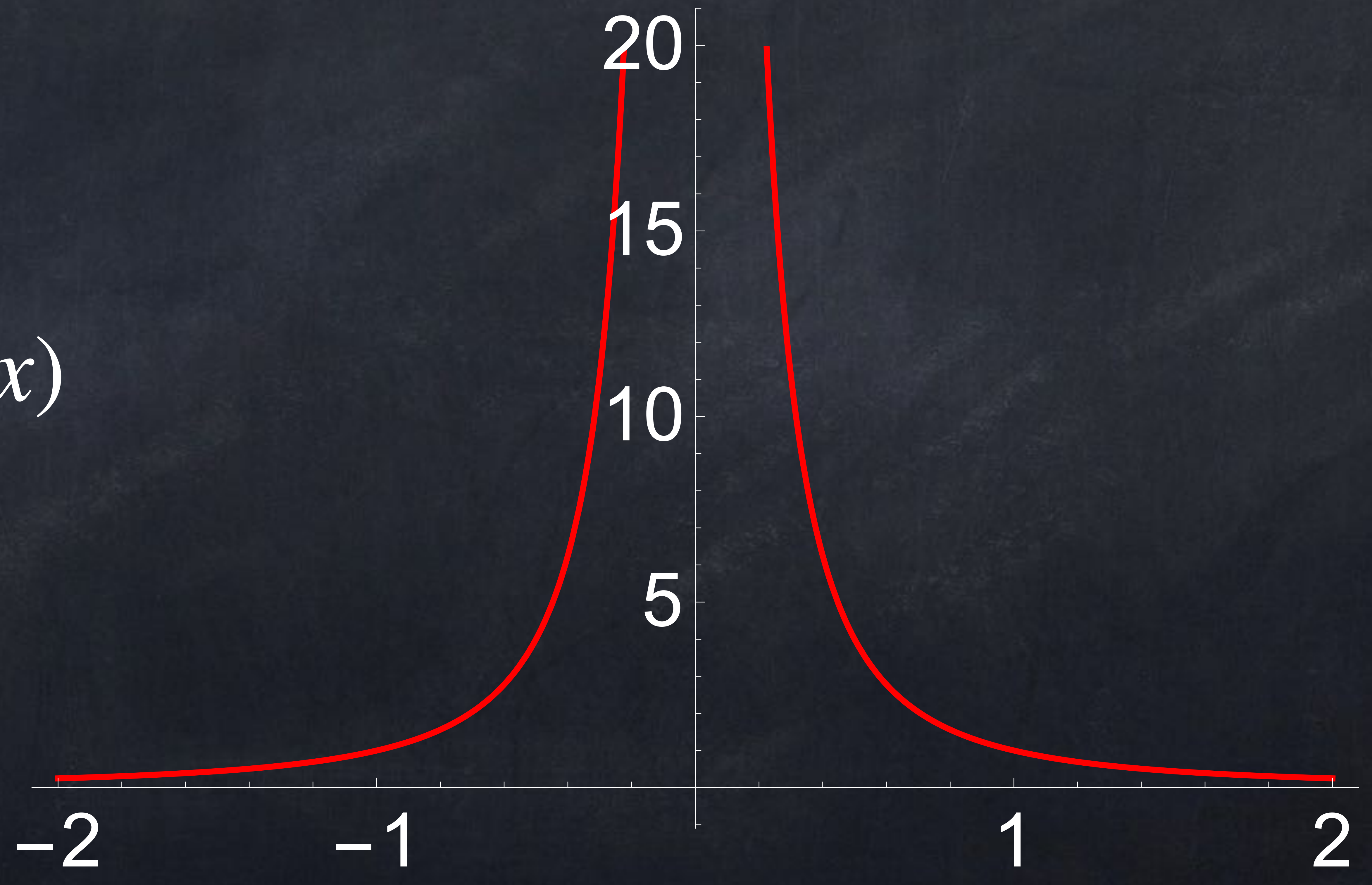
Depending on what formulas are causing 0 or $\pm\infty$ to appear, limits with these patterns can have many different values.

Infinite Limits

Sometimes the limit as x approaches some finite point will be ∞ or $-\infty$.

For example, $\lim_{x \to 0} \frac{1}{x^2} = \infty$.

This means that for values of x very close to 0, the values of $f(x)$ are all extremely large.



Infinite Limits

Sometimes the limit as x approaches some finite point will be ∞ or $-\infty$.

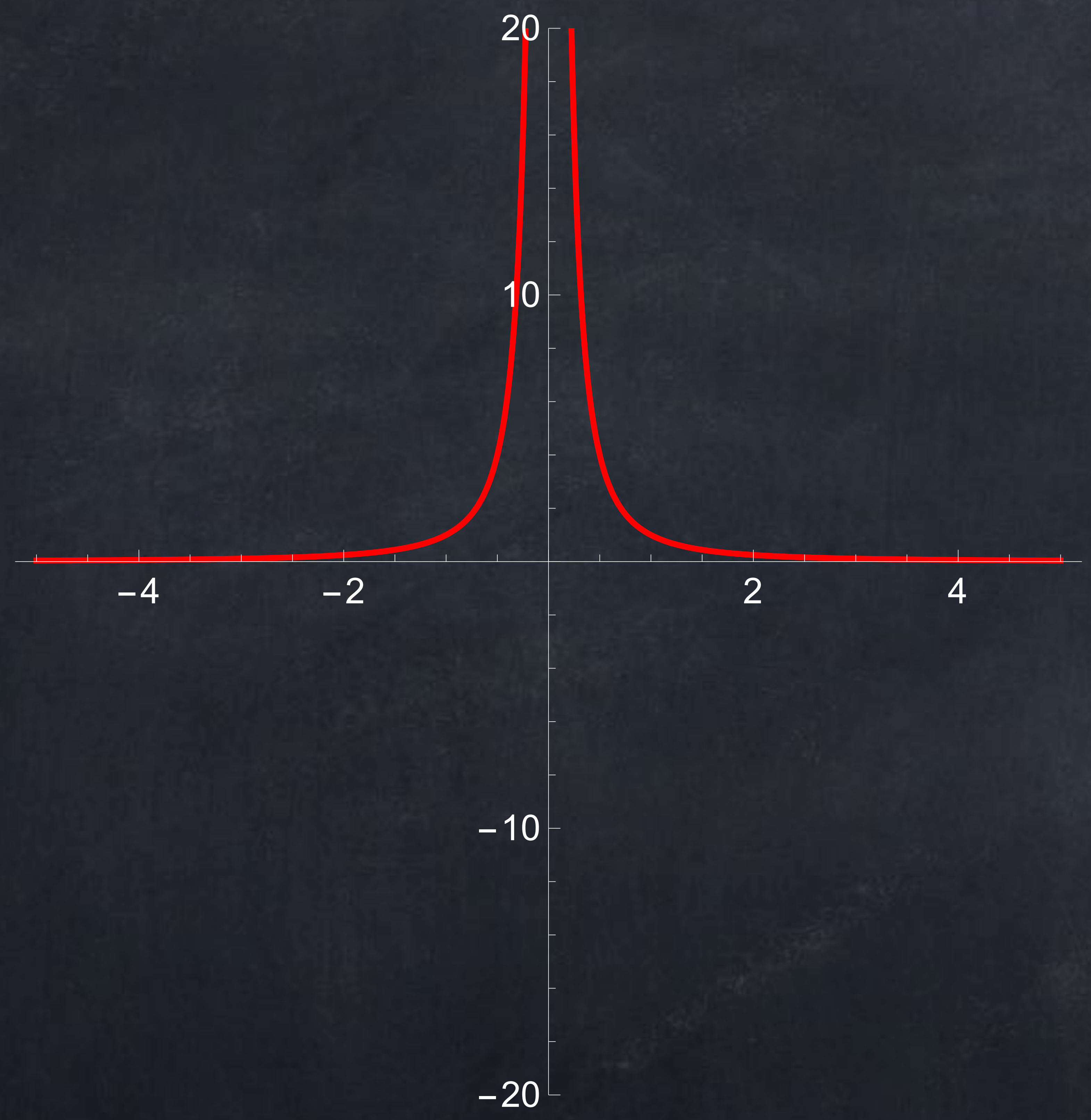
For example, $\lim_{x \to 0} \frac{1}{x^2} = \infty$.

Official definitions (we won't use these):

- $\lim_{x \to a} f(x) = \infty$ means that for any $M > 0$ there exists $\delta > 0$ such that
  if $|x - a| < \delta$ then $f(x) > M$.

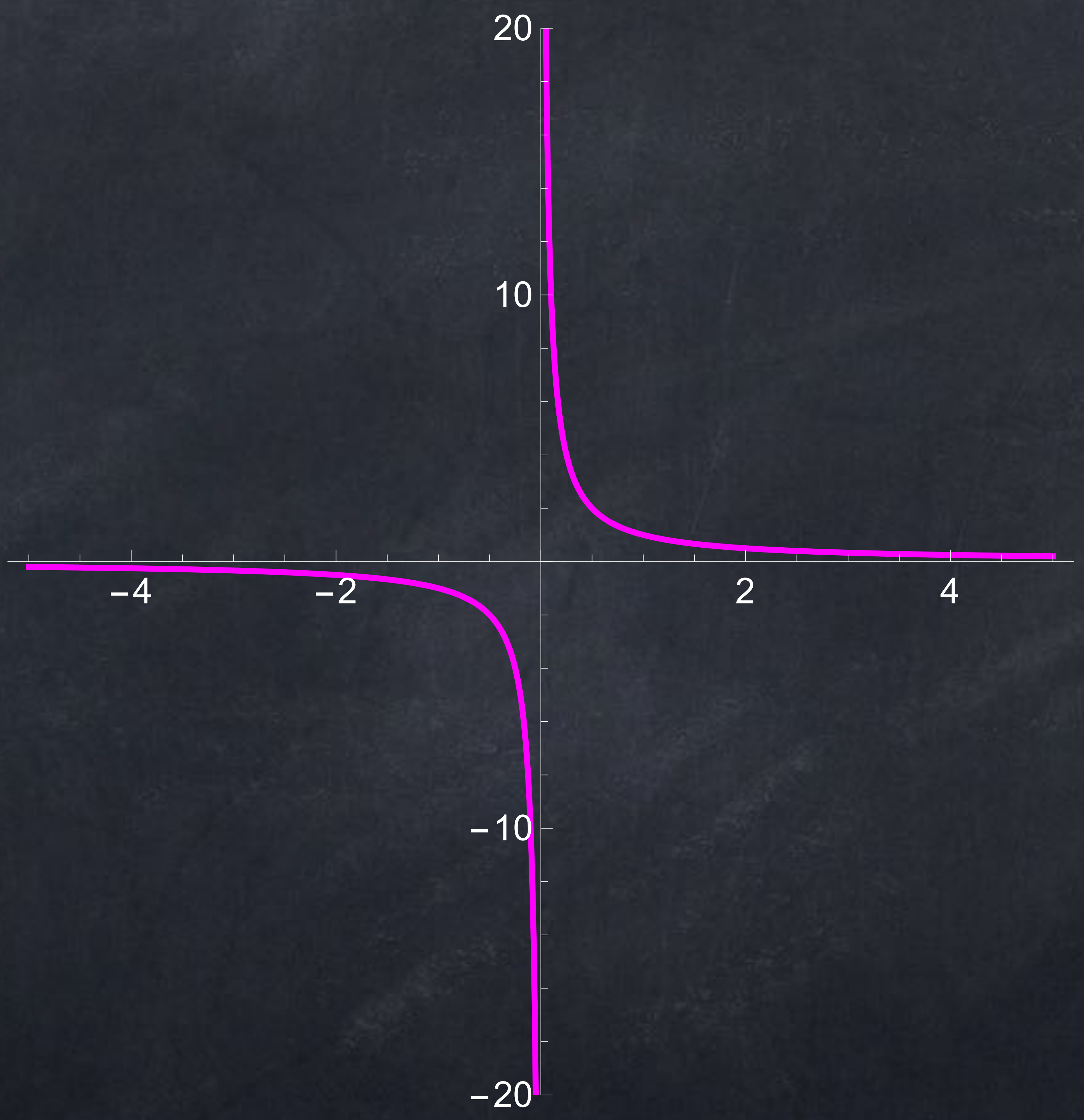
- $\lim_{x \to a} f(x) = -\infty$ means that for any $M > 0$ there exists $\delta > 0$ such that
  if $|x - a| < \delta$ then $f(x) < -M$.

$$y = \frac{1}{x^2}$$



$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

$$y = \frac{1}{x}$$



$$\lim_{x \to 0} \frac{1}{x} \text{ doesn't exist}$$

One-sided Limits

We write

$$\lim_{x \to a^-} f(x)$$

for the "**limit as x approaches a from the left**" or "… from below". This means we only look at x values that are less than a.

Similarly,

$$\lim_{x \to a^+} f(x)$$

means the "**limit as x approaches a from the right**" or "… from above", where we only look at x values that are more than a.

One-sided Limits

We write

$$\lim_{x \to a^-} f(x)$$

for the "limit as x approaches a from the left." This means we only look at x values that are less than a.

Example: $\lim_{x \to 0^-} x\sqrt{1+\frac{1}{x^2}} = -1.$

x	-0.1	-0.05	-0.01	-0.001	-0.0001
f(x)	-1.00499	-1.001249	-1.00005	-1.0000005	-1.00000001

One-sided limits

We write

$$\lim_{x\to a^-} f(x)$$

for the "limit as x approaches a from the left." This means we only look at x values that are less than a.

Example: $\displaystyle\lim_{x\to 0^-} x\sqrt{1+\frac{1}{x^2}} = -1.$

$$\lim_{x\to 0^+} x\sqrt{1+\frac{1}{x^2}} = 1.$$

One-sided Limits

Note: writing

$$4^+$$

by itself does not mean anything (like $\sqrt{\ }$ or $|\quad|$ alone). This should only be written as part of a limit:

$$\lim_{x \to 4^+} f(x).$$

Some books use $\lim_{x \nearrow 4} f(x)$ and $\lim_{x \searrow 4} f(x)$ instead of $\lim_{x \to 4^-} f(x)$ and $\lim_{x \to 4^+} f(x)$.

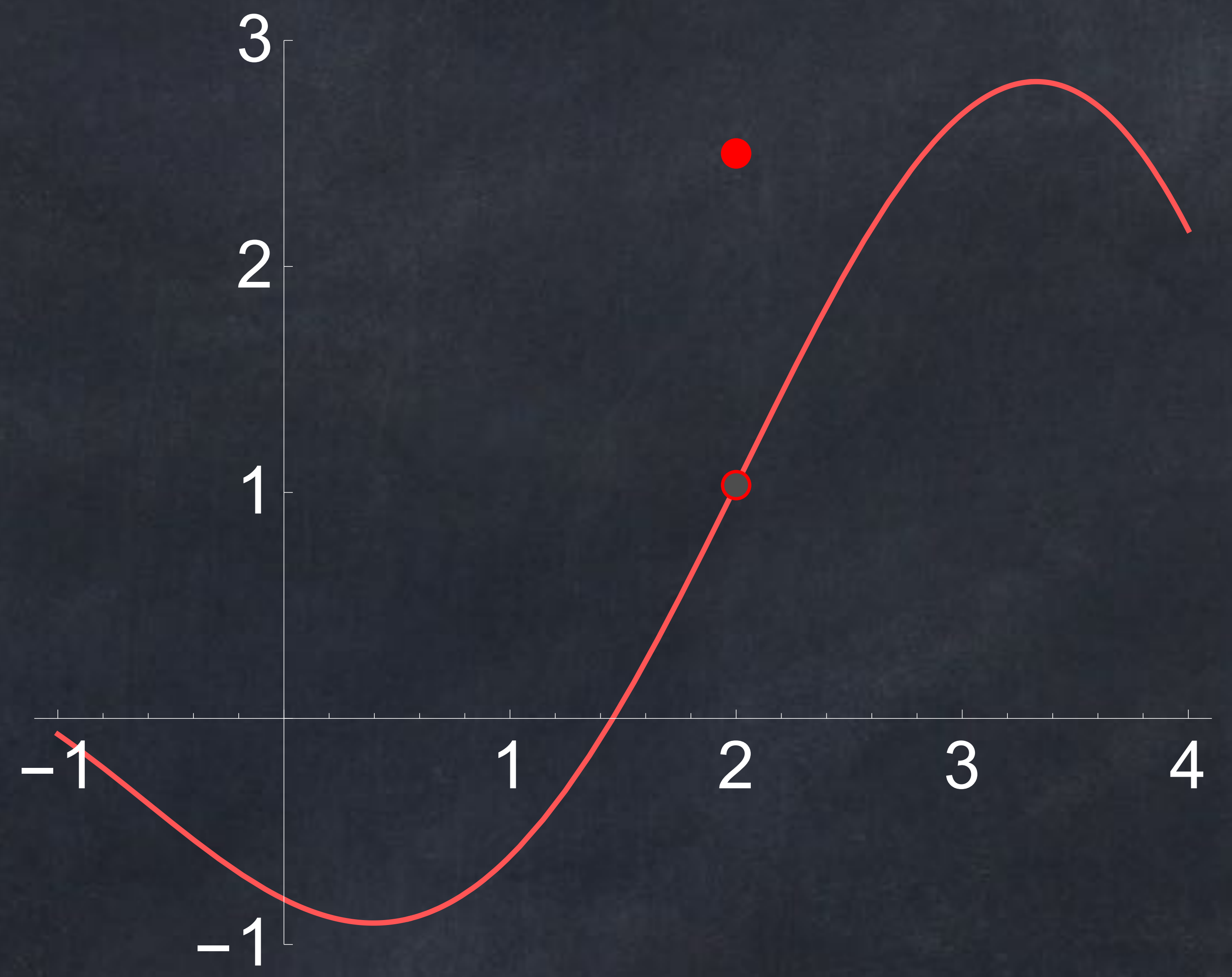
One-sided limits are related to standard limits in the following way:

> If $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ have different values, or if at least one of them does not exist, then $\lim_{x \to a} f(x)$ does not exist.

Logically, this also means that

- if $\lim_{x \to a} f(x)$ exists then $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$.

It's possible for $\lim_{x \to a} f(x)$ to be completely unrelated to the value $f(a)$.



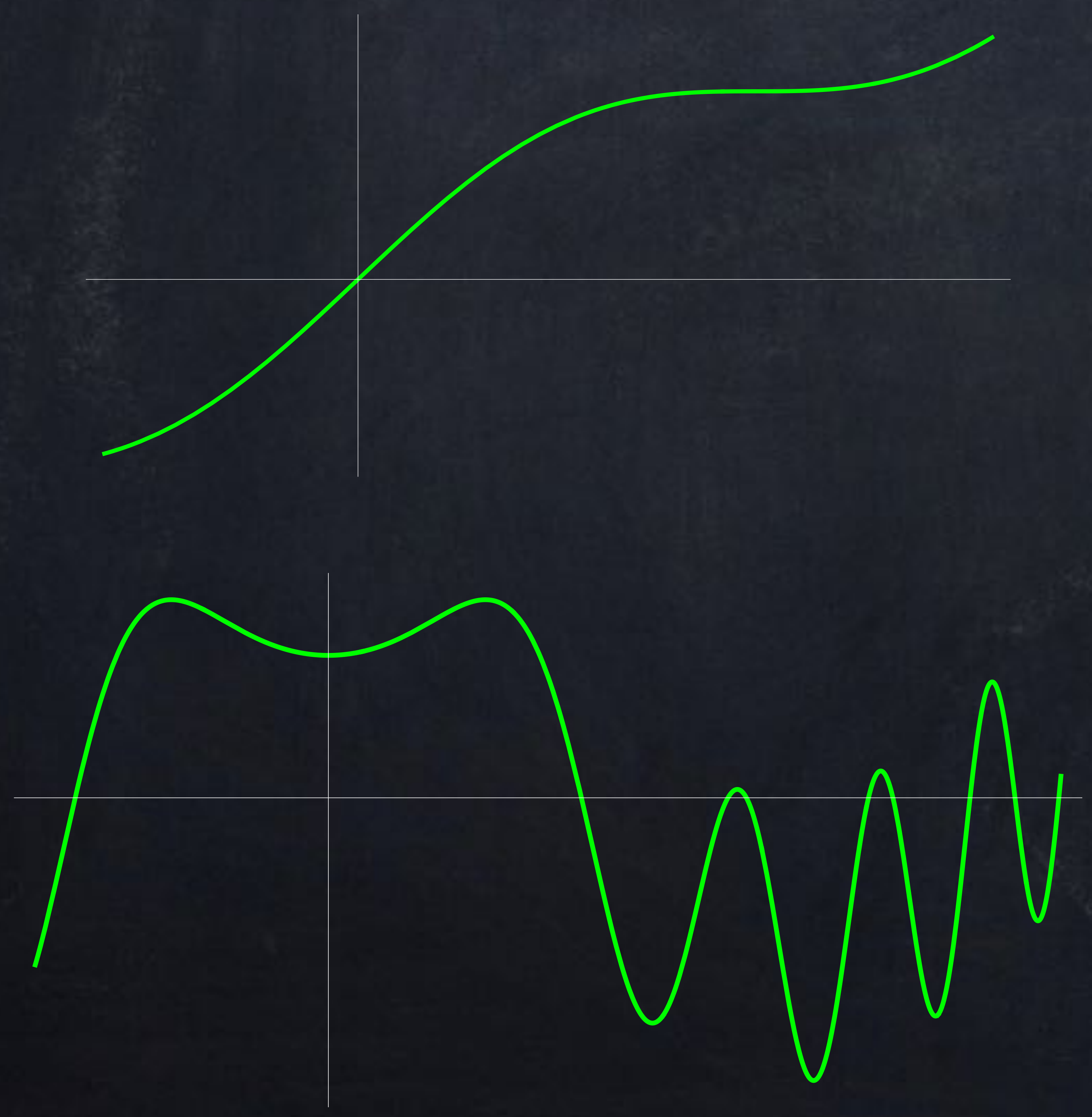
But for many functions we *can* calculate limits just by finding the value of the function at the point:

$$\lim_{x \to 7} \frac{x^2 + 1}{x^2 - 2} = \frac{7^2 + 1}{7^2 - 1} = \frac{50}{47}.$$

Continuity

Informally, a **continuous function** is one with no holes, jumps, or asymptotes in its graph. This means you could draw its graph without picking up your pen or pencil.

Continuous



Not continuous

Continuity

Let $f(x)$ be a function and let p be a number.

We say "**f is continuous at p**" if all of these are true:

1. $f(p)$ is defined,
2. $\lim_{x \to p} f(x)$ exists,
3. $\lim_{x \to p} f(x) = f(p)$.

If any of these is false, f is **discontinuous**.

What can discontinuity look like?

Types of discontinuities

The graph $y = f(x)$ has a **jump** at $x = a$ if

1. $\lim_{x \to a^-} f(x)$ is finite, and
2. $\lim_{x \to a^+} f(x)$ is finite, and
3. $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$.

The graph $y = f(x)$ has a **hole** at $x = a$ if

1. $\lim_{x \to a} f(x)$ is finite and
2. $f(a)$ is not defined or $f(a) \neq \lim_{x \to a} f(x)$.

Types of discontinuities

The vertical line $x = a$ is a **vertical asymptote** of the graph $y = f(x)$ if *any* of the following are true:

$$\lim_{x \to a^-} f(x) = -\infty \quad \text{or} \quad \lim_{x \to a^+} f(x) = -\infty \quad \text{or}$$

$$\lim_{x \to a^-} f(x) = +\infty \quad \text{or} \quad \lim_{x \to a^+} f(x) = +\infty.$$

Note: a horizontal asymptote ($\lim_{x \to \infty} = L$ or $\lim_{x \to -\infty} = L$) has nothing to do with discontinuity.

Task 1: What are the discontinuities of $f(x) = \dfrac{(x+7)(x-3)}{(x+2)(x-3)}$?

We will answer this next week.