Analysis 2

21 May 2024

Warm-up: Simplify $\dfrac{2(x+h)^2-18}{h}$ assuming $h \neq 0$.

Continuity

Last time

Let $f(x)$ be a function and let p be a number.

We say "**f is continuous at p**" if all of these are true:

1. $f(p)$ is defined,
2. $\lim_{x \to p} f(x)$ exists,
3. $\lim_{x \to p} f(x) = f(p)$.

If any of these is false, f is **discontinuous**.

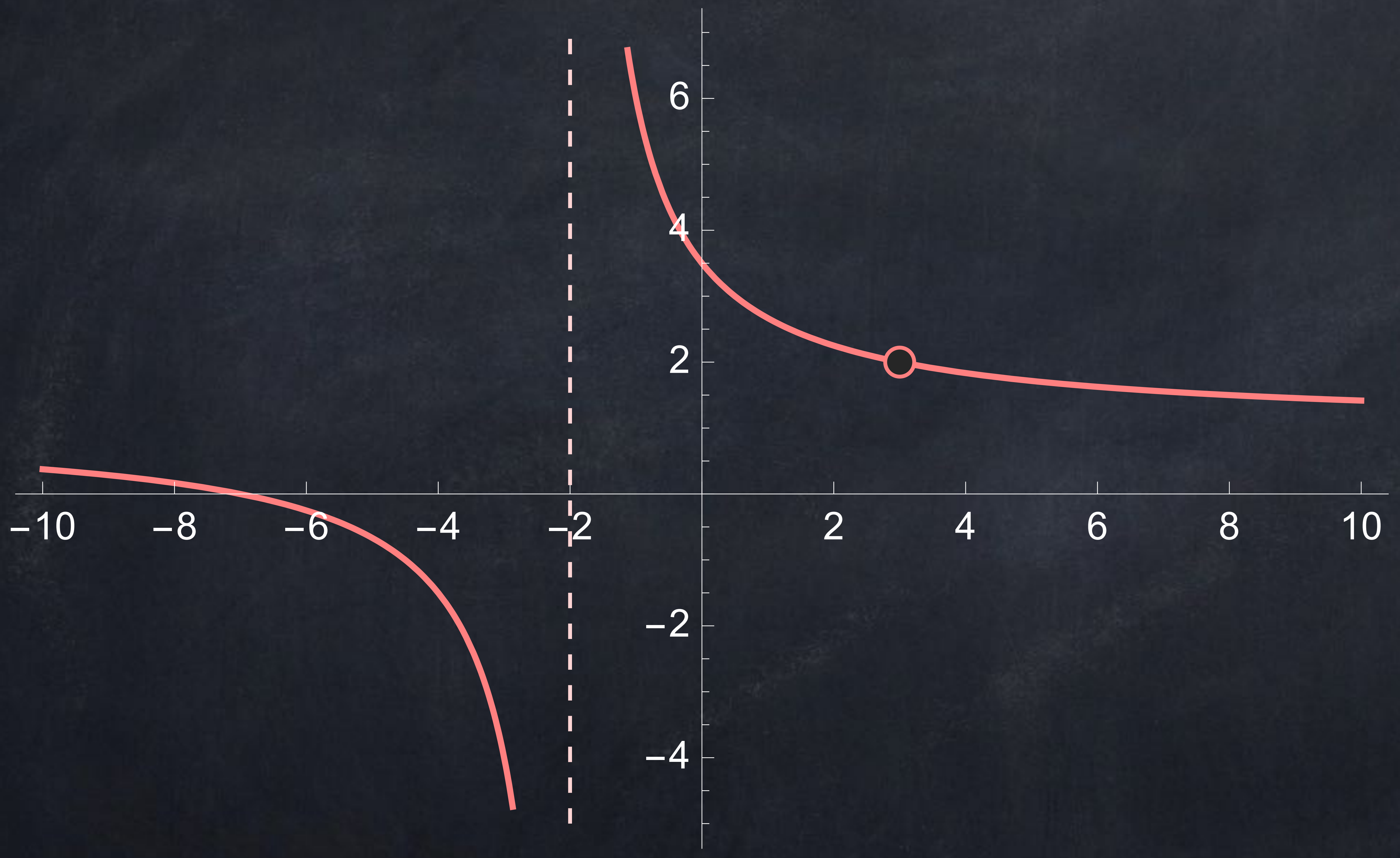
What can discontinuity look like?

Task 1(a): At what x-values is $f(x) = \dfrac{(x+7)(x-3)}{(x+2)(x-3)}$ discontinuous?

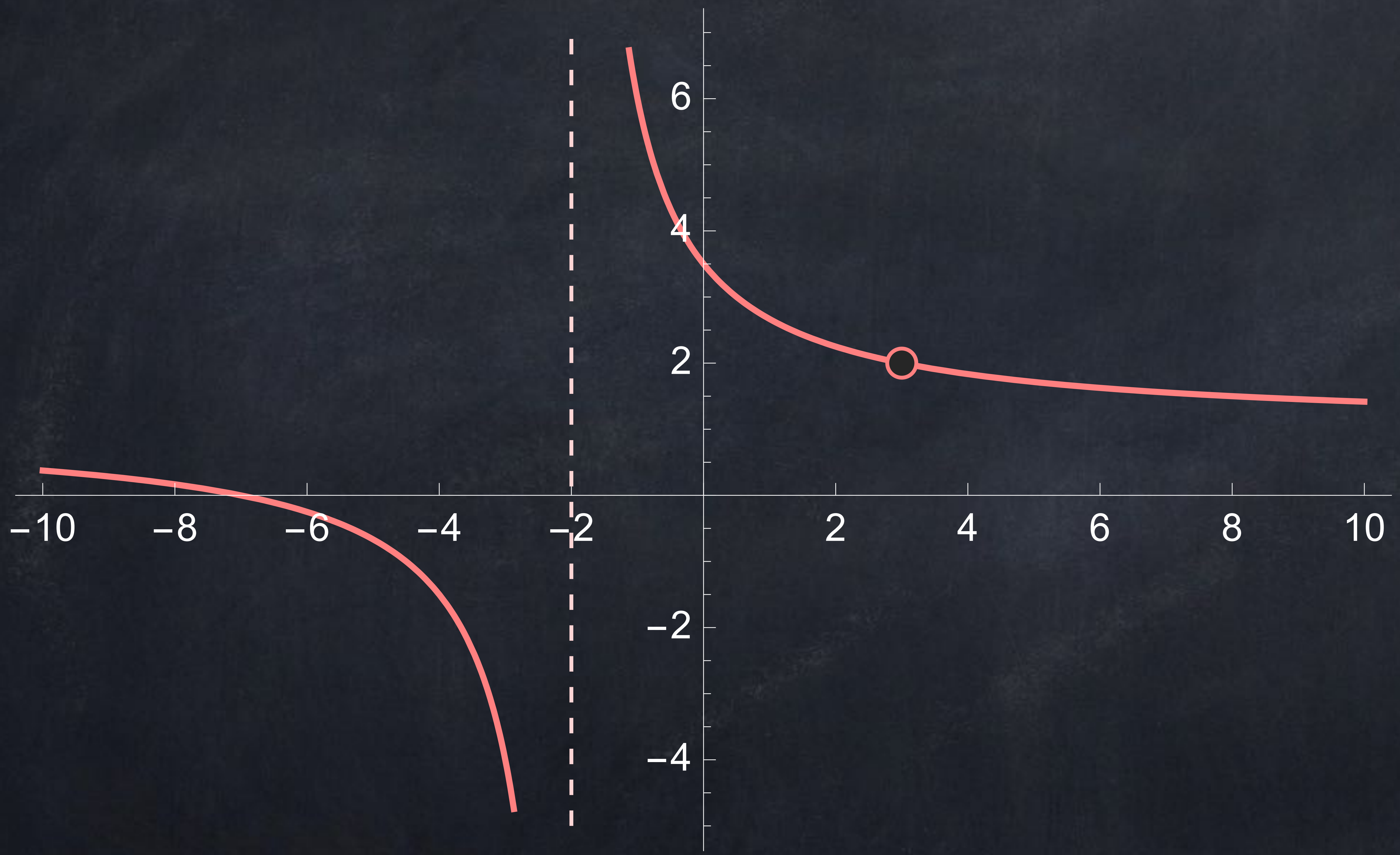
Continuous means

1. $f(p)$ is defined,
2. $\lim_{x \to p} f(x)$ exists,
3. $\lim_{x \to p} f(x) = f(p)$.

Task 1(b): *Describe* the discontinuities of $f(x) = \dfrac{(x+7)(x-3)}{(x+2)(x-3)}$.



Task 1(c): For the function $f(x) = \dfrac{(x+7)(x-3)}{(x+2)(x-3)}$, find…



$$\lim_{x \to -2^-} f(x)$$

$$\lim_{x \to -2^+} f(x)$$

$$\lim_{x \to -2} f(x)$$

$$\lim_{x \to 3^-} f(x)$$

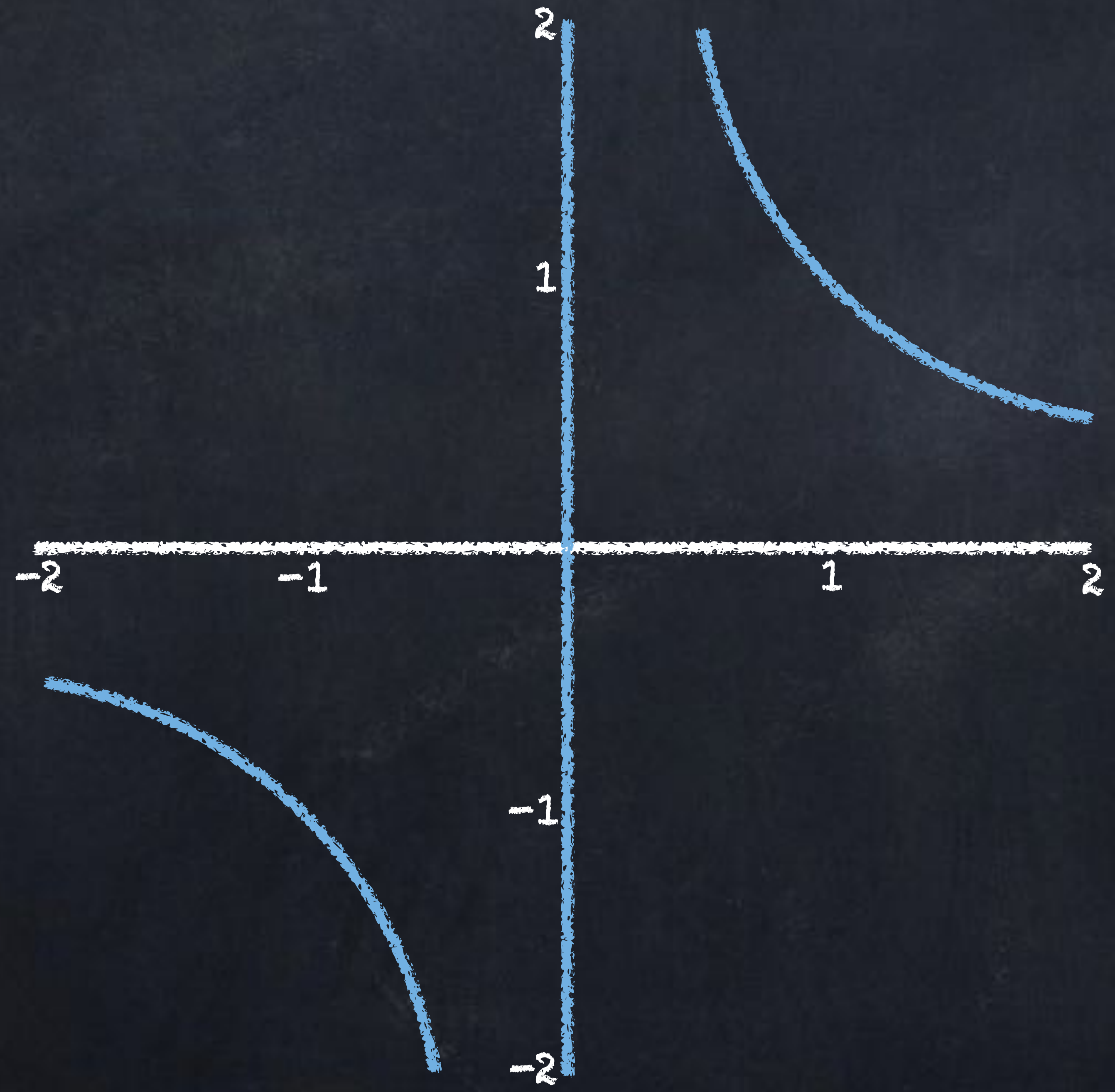
$$\lim_{x \to 3^+} f(x)$$

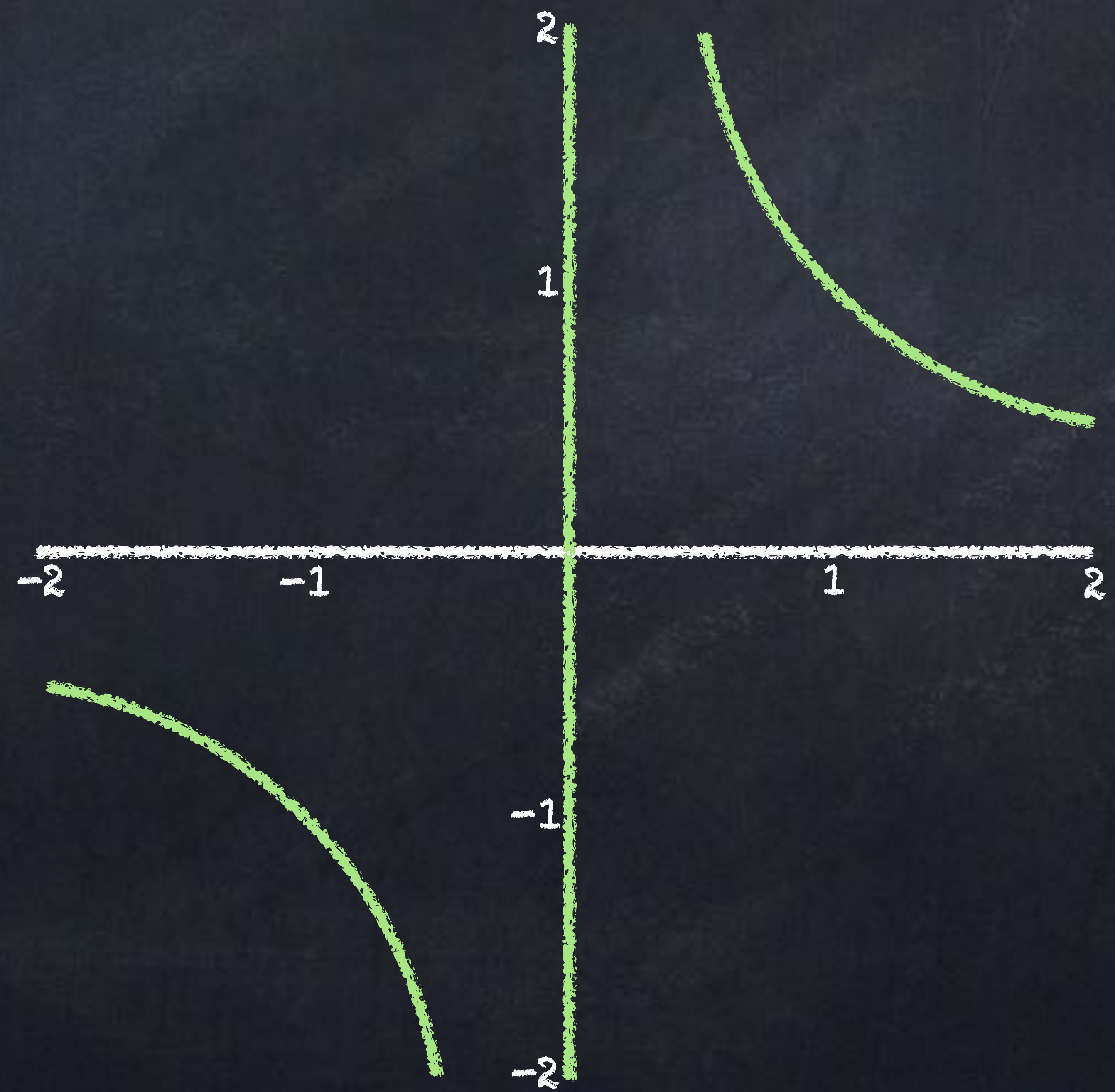
$$\lim_{x \to 3} f(x)$$

Why talk about limits when we can just ask a computer to graph a function?

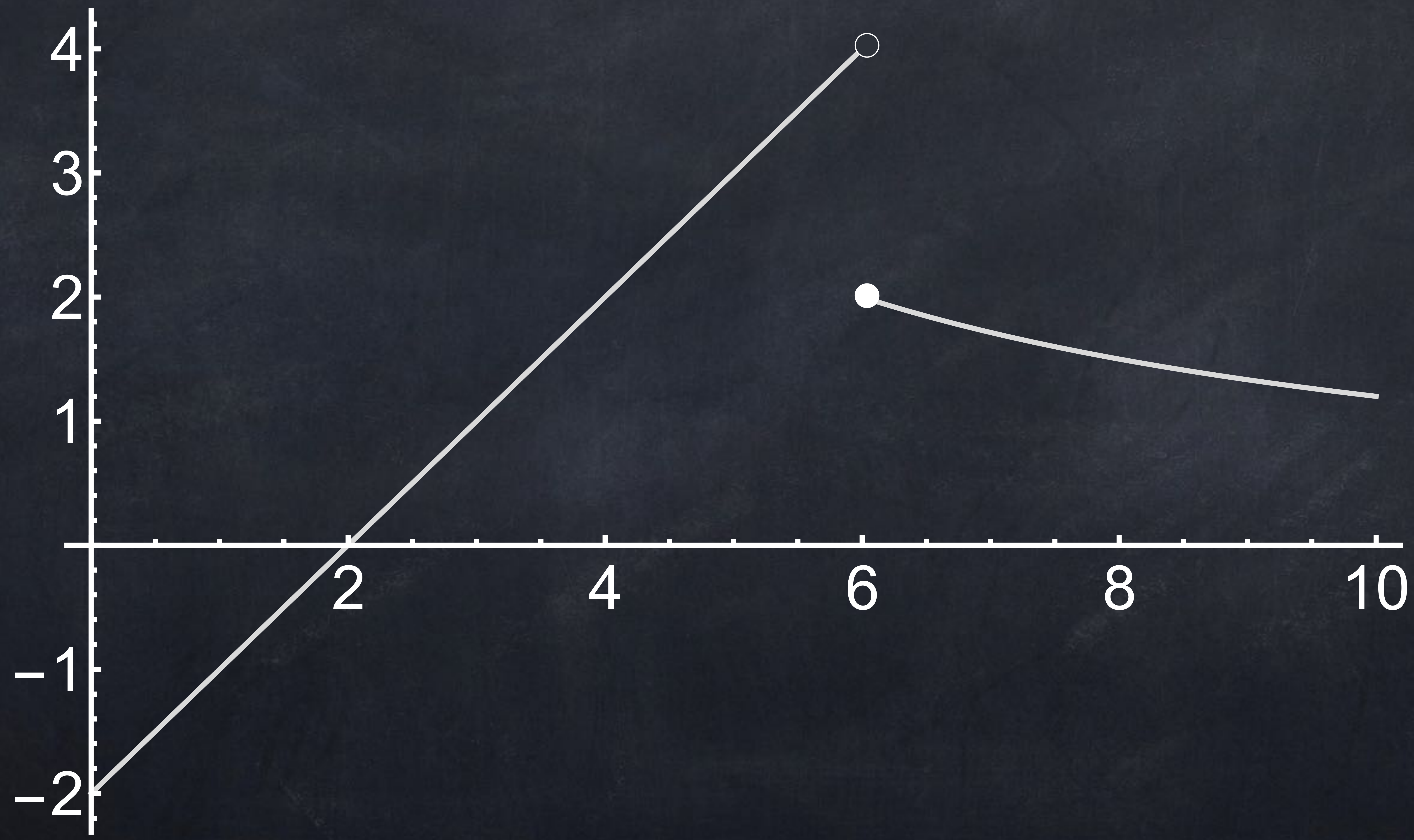
$$y = \frac{100x^2 + 100}{100x^3 + 100x}$$

$$y = \frac{100x^2}{100x^3 + x}$$





Task 2: What is $\lim_{x \to 6^-} f(x)$ for the function below? What is $\lim_{x \to 6} f(x)$?



Continuity

Our previous definition describes when a function is *continuous at a point*.

We say "**f is continuous on the interval $[a, b]$**" if it is continuous at all points p for which $a \leq p \leq b$.

- We can also talk about open intervals (a, b), semi-open intervals $[a, b)$ or $(a, b]$, and infinite intervals like $[a, \infty)$.

We say "**f is continuous everywhere**" or "**f is continuous**" or "**f is a continuous function**" if it is continuous at all points.

Limit properties

If the limits all exist and are finite, then

- $$\lim_{x\to a}\Big(f(x)+g(x)\Big) = \Big(\lim_{x\to a} f(x)\Big) + \Big(\lim_{x\to a} g(x)\Big),$$
- $$\lim_{x\to a}\Big(f(x)\cdot g(x)\Big) = \Big(\lim_{x\to a} f(x)\Big)\Big(\lim_{x\to a} g(x)\Big),$$
- $$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$ if $\lim_{x\to a} g(x) \neq 0$,
- $$\lim_{x\to a} f\big(g(x)\big) = f\Big(\lim_{x\to a} g(x)\Big)$$ if f is a ~~"nice"~~ **continuous** function.

These all work for $x \to \infty$, $x \to -\infty$, $x \to a^{+}$, and $x \to a^{-}$ too!

Continuity

Important examples:

- Any polynomial is continuous.
  - This includes constant functions.
- $\sqrt[n]{x}$ is continuous if n is odd.
- $\sqrt[n]{x}$ is continuous on $[0,\infty)$ if n is even.
- $\sin(x)$ and $\cos(x)$ are continuous.
- e^x and a^x are continuous.
- $\ln(x)$ and $\log_b(x)$ are continuous on $(0,\infty)$.

You can use all of these without giving any proofs or reasons.

Continuity

Fact: if $f(x)$ and $g(x)$ are both continuous, then the

- **sum** $f(x) + g(x)$,
- **difference** $f(x) - g(x)$,
- **product** $f(x) \cdot g(x)$, and
- **composition** $f(g(x))$

are all continuous.

- If $g(x)$ is never 0 then the **quotient** $\dfrac{f(x)}{g(x)}$ is also continuous.

Continuous means

1. $f(p)$ is defined,
2. $\lim_{x \to p} f(x)$ exists,
3. $\lim_{x \to p} f(x) = f(p)$.

Example: $\lim_{x \to 4} \dfrac{6^{\sin(\pi x)}}{1 + x^2} = \dfrac{6^{\sin(4\pi)}}{1 + 4^2} = \dfrac{1}{17}$.

It's common to see limits that look like "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$" if you just plug in the value. There is a nice trick we can use for these:

L'Hôpital's Rule

also spelled L'Hospital

If f and g are differentiable near $x = a$, and $\lim_{x \to a} \frac{f(x)}{g(x)}$ exists, and either

- $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$, or
- $\lim_{x \to a} f(x) = \pm\infty$ and $\lim_{x \to a} g(x) = \pm\infty$, ← technically four cases (++, +−, −+, −−)

then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$.

Also true for $\lim_{x \to a^+}$ and $\lim_{x \to a^-}$. Note that $\frac{f}{g} \neq \frac{f'}{g'}$; it is only the limits that are equal.

L'H: For $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$, use $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$.

Task 1:

Find $\lim_{x\to 1} \frac{x^3 - x^2 + x - 1}{x^4 - x^3 + 4x - 4}$.

Answer: $\frac{2}{5}$

Task 2. Calculate $\lim_{x \to 0} \dfrac{\sin(x)}{x}$.

L'H: For $\dfrac{0}{0}$ or $\dfrac{\pm\infty}{\pm\infty}$, use $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \lim_{x \to a} \dfrac{f'(x)}{g'(x)}$.

Use L'Hôpital 3 times.

Final answer: $\dfrac{-1}{6}$

Task 3. Find $\lim_{x \to 4} \dfrac{4x^2 - 28}{x^2 - 4}$.

L'H: For $\dfrac{0}{0}$ or $\dfrac{\pm\infty}{\pm\infty}$, use $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \lim_{x \to a} \dfrac{f'(x)}{g'(x)}$.

Answer: 3 (NOT 4)

Derivatives

What is $\frac{\mathrm{d}}{\mathrm{d}x}\left[x^5\right]$ at $x = 3$?

What is $\frac{\mathrm{d}}{\mathrm{d}x}\left[2^5\right]$ at $x = 3$?

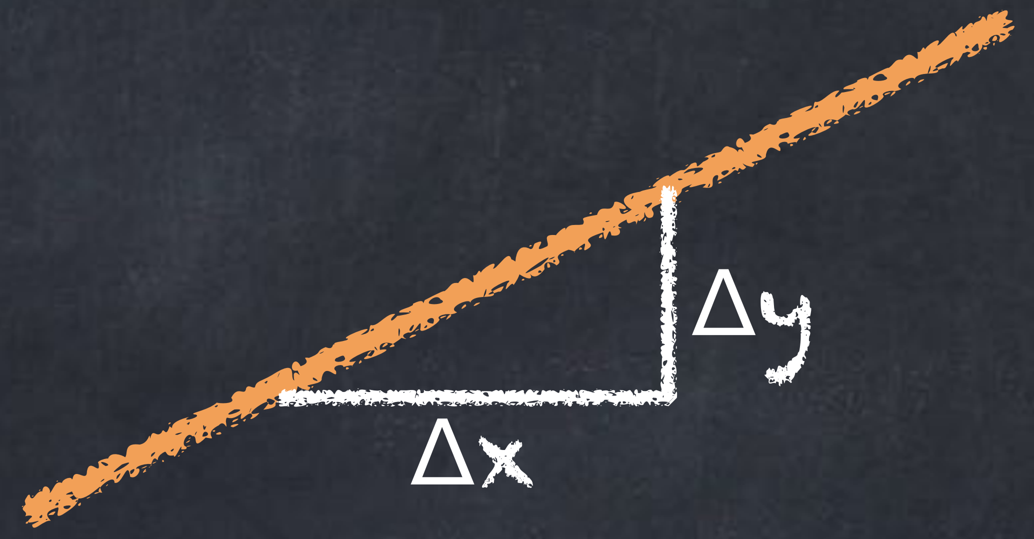
What is $\frac{\mathrm{d}}{\mathrm{d}x}\left[2^x\right]$ at $x = 3$?

What is $\frac{\mathrm{d}}{\mathrm{d}x}\left[x^x\right]$ at $x = 3$?

- How could we approximate $(x^x)'$ at $x = 2$ using only a basic calculator?

Derivatives

For a straight line, slope $= \dfrac{\text{rise}}{\text{run}} = \dfrac{\Delta y}{\Delta x}$.



For a curved graph, this fraction is not constant. It will change if we use a different place to start drawing a small triangle, and it will change if we keep the corner the same and make Δx bigger or smaller.

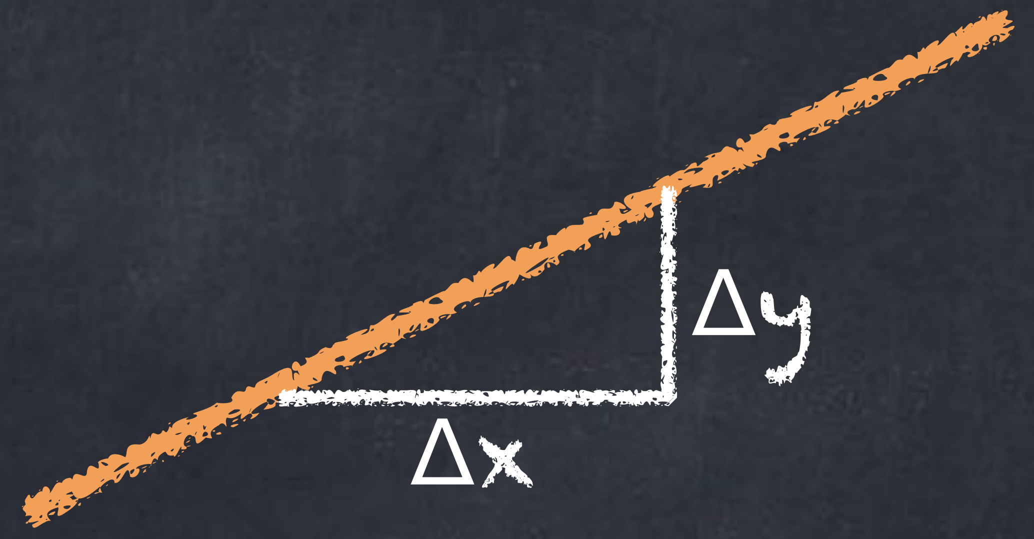
- How could we *approximate* $(x^x)'$ at $x = 3$ using only a basic calculator?

$$\frac{3.01^{3.01} - 27}{0.01} \approx 57.31 \qquad \frac{3.0001^{3.0001} - 27}{0.0001} \approx 56.669$$

The correct derivative is about 56.663.

Derivatives

For a straight line, slope $= \dfrac{\text{rise}}{\text{run}} = \dfrac{\Delta y}{\Delta x}$.



For a curved graph, this fraction is not constant. It will change if we use a different place to start drawing a small triangle, and it will change if we keep the corner the same and make Δx bigger or smaller.

To find the exact slope for a curved graph, we need $\displaystyle\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$.

- This is why we also write $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ for y'.

Limits and Derivatives

Technically, $f'(x)$ is *defined* as

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x},$$

if this limit exists. Using this definition, for example,

- the derivative of $2x^3$ at $x = 5$ is $\lim_{\Delta x \to 0} \frac{2(5 + \Delta x)^3 - 2(5)^3}{\Delta x} = \cdots = 150$.

Of course, we know $(2x^3)' = 6x^2$ from the Power Rule, and it's faster to just calculate $6(5)^2 = 6 \cdot 25 = 150$. But the only way people found the Power Rule originally was to do $\lim_{\Delta x \to 0} \frac{(x + \Delta x)^n - x^n}{\Delta x}$.

If you have the paper

$9x^4$

then you should find the person with $36x^3$.

Anti-derivatives

The words “anti-derivative” and “integral” are closely related, but for today we will only talk about anti-derivatives.

- The derivative of $9x^4$ is $36x^3$.
- The function $9x^4$ is an **anti-derivative** of $36x^3$.
- The function $9x^4 - 7$ is *also* an anti-derivative of $36x^3$. It’s not unique.

$$? \longrightarrow x^3 \longrightarrow 3x^2 \longrightarrow 6x$$

$$\frac{1}{4}x^4 \longrightarrow x^3 \longrightarrow 3x^2 \longrightarrow 6x$$

$$\frac{1}{4}x^4 \longrightarrow x^3 \longrightarrow 3x^2 \longrightarrow 6x$$

$$-\cos(x) \longrightarrow \sin(x) \longrightarrow \cos(x)$$

$$? \longrightarrow -x^{-1} \longrightarrow x^{-2} \longrightarrow -2x^{-3}$$

So far this semester we have avoided using logarithms. Now we cannot ignore them!

$$-\ln(x) \longrightarrow -x^{-1} \longrightarrow x^{-2} \longrightarrow -2x^{-3}$$