Analysis 1, Summer 2024

List 1

Domain, function composition, derivative rules

20. Label each of the following expressions as "a sum", "a difference", "a product", "a quotient", or "a composition".
    (a) $x^2 + 7$ sum
    (b) $(x+7)^2$ composition or product
    (c) $\sin(x+7)$ composition
    (d) $\dfrac{(x-1)^3}{e^x} - \dfrac{1}{x+8}$ difference or sum
    (e) $\dfrac{5\sin(2x)}{e^{(\sin(x))^3}}$ quotient or product if re-written as $5\sin(2x)e^{-(\sin x)^3}$
    (f) $\sqrt{\dfrac{1}{x} + \dfrac{1}{x^2}}$ composition
    (g) $\sin(\sqrt{x}) + \sqrt[3]{\sin(x)}$ sum

21. Give the composition $f(g(x))$ for the functions $f(x) = x^2 e^x$ and $g(x) = 8x - 3$ (this can also be written as $f \circ g$). $(8x-3)^2 e^{8x-3}$

The **natural domain** for a function given by a formula is the largest set of (real) numbers on which the formula is defined.

22. Give the natural domain of each of the following functions:
    (a) $f(x) = \dfrac{18+x}{5-x}$ $x \neq 5$ or $(-\infty, 5) \cup (5, \infty)$
    (b) $g(x) = \dfrac{\sqrt{18+x}}{5-x}$ $x \geq -18 \wedge x \neq 5$ or $[-18, 5) \cup (5, \infty)$
    (c) $k(x) = \dfrac{18+x}{\sqrt{5-x}}$ $x < 5$ or $(-\infty, 5)$
    (d) $f(x) = \sqrt{\dfrac{18+x}{5-x}}$ $-18 \leq x < 5$ or $[-18, 5)$
    (e) $g(x) = e^{7x+2}$ all x or $(-\infty, \infty)$ or $\mathbb{R}$
    (f) $k(x) = \dfrac{1}{\sqrt{x}} + \dfrac{1}{2x-7}$ $(0, \frac{7}{2}) \cup (\frac{2}{7}, \infty)$

23. What is the natural domain of $f(x) = \dfrac{2x-7}{2x^2+9x+4}$? $(-\infty, -4) \cup (-4, -\frac{1}{2}) \cup (\frac{1}{2}, \infty)$

24. What is the natural domain of $g(x) = \dfrac{2x-7}{\sqrt{2x^2+9x+4}}$? $(-\infty, -4) \cup (\frac{1}{2}, \infty)$

25. Find the natural domain of
    (a) $\sin(x-4)$ $\mathbb{R}$
    (b) $2\sin(\sqrt{x-4})$ $[4, \infty)$
    (c) $3\sin(\sqrt{e^x})$ $\mathbb{R}$
    (d) $\dfrac{4}{\sin(x)}$ $\{x \in \mathbb{R} : x \neq k\pi, k \in \mathbb{Z}\}$

26. Give an example of a function whose natural domain is $[0,1) \cup (1,\infty)$.
There are many possibilities. In order to get $x \geq 0$ the easiest thing is $\sqrt{x}$. In order to remove $x = 1$ the easiest is $\dfrac{1}{x-1}$. Both $\boxed{\dfrac{\sqrt{x}}{x-1}}$ and $\boxed{\sqrt{x} + \dfrac{1}{x-1}}$ are simple correct answers.

For a function $f(x)$ and a number a, the **derivative of f at a**, written $f'(a)$, is official defined as

$$f'(a) \quad = \quad \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} \quad = \quad \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

and is equal to the slope of the tangent line to $y = f(x)$ at the point $(a, f(a))$.
We can also think of the **derivative of f** as a new function, written $f'(x)$ or $\frac{\mathrm{d}f}{\mathrm{d}x}$.

The Power Rule: If p is a constant then $(x^p)' = p\,x^{p-1}$.

The Constant Multiple Rule: If c is a constant then

$$(c\,f)' = c\,f' \qquad \big(c\,f(x)\big)' = c\,f'(x) \qquad \tfrac{\mathrm{d}}{\mathrm{d}x}\big[c\,f\big] = c\,\tfrac{\mathrm{d}f}{\mathrm{d}x} \qquad D\big[c\,f\big] = c\,D[f]$$

(these are four ways of writing exactly the same fact).

The Sum Rule: $(f+g)' = f' + g'$.

27. All parts of this task have exactly the same answer! Answer: $\boxed{14x^6}$
    (a) Find $f'(x)$ for the function $f(x) = 2x^7$.
    (b) Give f' if $f = 2x^7$.
    (c) Find y' for $y = 2x^7$.
    (d) Compute $\frac{\mathrm{d}f}{\mathrm{d}x}$ for the function $f(x) = 2x^7$.
    (e) Compute $\frac{\mathrm{d}y}{\mathrm{d}x}$ for $y = 2x^7$.
    (f) Give the derivative of $2x^7$ with respect to x.
    (g) Find the derivative of $2x^7$.
    (h) Calculate $\frac{\mathrm{d}}{\mathrm{d}x}2x^7$. (i) Calculate $\big(2x^7\big)'$. (j) Calculate $D\big[2x^7\big]$.
    (k) Differentiate $2x^7$ with respect to x.
    (ℓ) Differentiate $2x^7$.

28. Differentiate $x^5 + \frac{2}{9}x^3 + \sqrt{3x} + \dfrac{x^{10}}{\sqrt{x}}$. $\boxed{5x^4 + \dfrac{2}{3}x^2 + \dfrac{\sqrt{3}}{2\sqrt{x}} + \dfrac{19}{2}x^{17/2}}$

29. Differentiate $\big(x + \sqrt{x}\big)^2$. $\boxed{2x + 3\sqrt{x} + 1}$ or $2\big(x + \sqrt{x}\big)\big(1 + \frac{1}{2\sqrt{x}}\big)$

☆30. Differentiate $\big(x + \sqrt{x}\big)^{100}$. $\boxed{100\big(x + \sqrt{x}\big)^{99}\big(1 + \frac{1}{2\sqrt{x}}\big)}$

31. For each of the functions below, can the Power Rule and/or Constant Multiple Rule (along with maybe some algebra) be used to find the derivative? If so, give the derivative.

(a) $2x^6$ $\boxed{\text{Yes: } 12x^5}$

(b) $2\sqrt{x}$ $\boxed{\text{Yes: } x^{-1/2}}$ or $\dfrac{1}{\sqrt{x}}$

(c) $\sqrt{5x}$ $\boxed{\text{Yes: } \frac{\sqrt{5}}{2}x^{-1/2}}$

(d) x^π $\boxed{\text{Yes: } \pi x^{\pi-1}}$

(e) $x^{\sin x}$ $\boxed{\text{No}}$

(f) x^x $\boxed{\text{No!}}$

(g) $\dfrac{x^{10}+3}{x^6}$ $\boxed{\text{Yes: } 4x^3 - 18x^{-7}}$

(h) $\dfrac{x^6}{x^{10}+3}$ $\boxed{\text{No}}$

(i) $\sqrt{9x^2+6x+4}$ $\boxed{\text{No}}$

(j) $\sqrt{9x^2}+\sqrt{4x}$ $\boxed{\text{Yes: } 3+\dfrac{1}{\sqrt{x}}}$

(k) $\sqrt{9x^2+12x+4}$ $\boxed{\text{Yes: } 3}$

32. Calculate both $f(5)$ and $f'(5)$ for $f(x)=x^3$. $f(5)=(5)^3=\boxed{125}$, and $f'(5)=4(5)^2=\boxed{75}$.

33. Give an equation for the tangent line to x^3 through the point $(5,125)$. $\boxed{y=125+75(x-5)}$ or $y=75x-250$

$\boxed{y=30+7(x-212)}$ or $\boxed{y=7x+2}$. (Since $y=7x+2$ is a straight line, the tangent line to it—at any point—is exactly itself.)

34. Give an equation for the tangent line to $y=x^3-x$ at $x=2$. $\boxed{y=6+11(x-2)}$ Other formats, such as $y=11x-16$, may also be correct.

35. Give an equation for the tangent line to $7x+2$ through the point $(30,212)$.

$\boxed{y=30+7(x-212)}$ or $\boxed{y=7x+2}$. (Since $y=7x+2$ is a straight line, the tangent line to it—at any point—is exactly itself.)

☆36. Find a line that is tangent to both $y=x^2+20$ and $y=x^3$. $\boxed{y=12x-16}$ is tangent to $y=x^2+20$ at $x=6$ and tangent to $y=x^3$ at $x=2$.

37. Give the derivative of each of the following functions.

(a) x^{7215} $\boxed{7215x^{7214}}$

(b) $5x^{100}+9x$ $\boxed{500x^{99}+9}$

(c) $2x^3-6x^2+10x+1$ $\boxed{6x^2-12x+10}$

(d) $3\sqrt{x}$ $\boxed{\frac{3}{2}x^{-1/2}}$ or $\boxed{\dfrac{3}{2\sqrt{x}}}$

(e) $\sqrt[3]{x}$ $\boxed{\frac{1}{3}x^{-2/3}}$ or $\boxed{\dfrac{1}{3\sqrt[3]{x^2}}}$

(f) $\sqrt{x}^3$ $\boxed{\frac{2}{3}x^{-1/3}}$ or $\boxed{\dfrac{2}{3\sqrt[3]{x}}}$

(g) 31 $\boxed{0}$

(h) $x + \frac{1}{x}$ $\boxed{1 - x^{-2}}$ or $\boxed{1 - \dfrac{1}{x^2}}$

(i) $\sqrt{x} + \frac{1}{\sqrt{x}}$ $\boxed{\frac{1}{2}x^{-1/2} + \frac{-1}{2}x^{-3/2}}$ or $\boxed{\dfrac{1}{2\sqrt{x}} - \dfrac{1}{2\sqrt{x^3}}}$

(j) $(3x+7)^2$ $\boxed{18x + 42}$ or $\boxed{6(3x+7)}$

38. Give an example of a function whose derivative is $7x^6 + 8x^3 + 9$. $\boxed{x^7 + 2x^2 + 9x}$ Adding any constant to this also gives a correct answer. This includes $x^7 + 2x^2 + 9x + 1$ and $x^7 + 2x^2 + 9x + \sqrt{37}$ and $x^7 + 2x^2 + 9x - 58$, etc.

39. If $f(x) = x^3 - x^2 - x$, for what values of x does $f(x) = 0$? $\boxed{x = \dfrac{1-\sqrt{5}}{2}, x = 0, x = \frac{1+\sqrt{5}}{2}}$

For what values of x does $f'(x) = 0$? $\boxed{\dfrac{-1}{4}, 0, \dfrac{1}{4}}$

The derivative of $\sin(x)$ is $\cos(x)$. The derivative of $\cos(x)$ is $-\sin(x)$. In symbols,

$$\frac{\mathrm{d}}{\mathrm{d}x}\big[\sin(x)\big] = \cos(x) \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}x}\big[\cos(x)\big] = -\sin(x).$$

40. Give the derivative of $5\sin(x) + \frac{2}{3}\cos(x) - x^3 + 9$. $\boxed{5\cos(x) - \frac{2}{3}\cos(x) - 3x^2}$

41. Give the derivative of each of the following:

(a) $\frac{1}{2}x^4 + 4\sin(x)$ $\rightarrow$ $\boxed{2x^3 + 4\cos(x)}$

(b) $2x^2 + 4\cos(x)$ $\rightarrow$ $\boxed{4x - 4\sin(x)}$

(c) $4x - 4\sin(x)$ $\rightarrow$ $\boxed{4 - 4\cos(x)}$

(d) $4 - 4\cos(x)$ $\rightarrow$ $\boxed{4\sin(x)}$

(e) $4\sin(x)$ $\rightarrow$ $\boxed{4\cos(x)}$

(f) $x^2 + (\sin x)^2 + (\cos x)^2$ $\rightarrow$ $\boxed{2x}$ because the original function is $x^2 + 1$

Product Rule: $(fg)' = fg' + f'g$, also written $\frac{\mathrm{d}}{\mathrm{d}x}[fg] = f\frac{\mathrm{d}g}{\mathrm{d}x} + \frac{\mathrm{d}f}{\mathrm{d}x}g$.

42. For each function below, state whether it is possible to find the derivative using only algebra, the Sum Rule, the Product Rule, and the derivatives of power and trig functions. If it is, give the derivative.

(a) 1 has derivative $\boxed{0}$. Note that $1 = x^0$, so we can use the Power Rule to get $0 \cdot x^{-1} = 0$ as its derivative. Alternatively, $1 = x \cdot x^{-1}$, so we can use the Power and Product Rules together to get $(x)(-x^{-2}) + (1)(x^{-1}) = -x^{-1} + x^{-1} = 0$ as the derivative.

(b) $6\sin(x)$ has derivative $6\cos(x)$. The Power Rule would describe this as $6(\sin(x))' + (6)'\sin(x) = 6\cos(x) + 0\sin(x)$, which is $6\cos(x)$.

(c) $\sqrt{16x}$ is equal to $4x^{1/2}$ and therefore has derivative $\boxed{2x^{-1/2}}$, or $\boxed{\frac{2}{\sqrt{x}}}$.

(d) $\sqrt{16\sin(x)}$ requires the Chain Rule

(e) $(x + \sqrt{7})^2$ is equal to $x^2 + 2\sqrt{7}x + 7$ and therefore has derivative $\boxed{2x + 2\sqrt{7}}$.

(f) 2^{x+7} is equal to $128 \cdot 2^x$, but differentiating this requires another rule (the Exponential Rule).

(g) $\dfrac{\cos(x)}{3}$ has derivative $\boxed{\frac{-1}{3}\sin(x)}$.

(h) $\dfrac{\cos(x)}{x}$ is equal to $x^{-1}\cos(x)$ and therefore has derivative $\boxed{-x^{-1}\sin(x) - x^{-2}\cos(x)}$, or $\boxed{-\dfrac{x\sin(x) + \cos(x)}{x^2}}$.

(i) $\dfrac{3}{\cos(x)}$ requires the Quotient Rule (or the Chain and Product Rules together).

43. Using the Product Rule, give the derivative of $\sqrt{x} \cdot \sin(x)$. $\boxed{\sqrt{x}\cos(x) + \dfrac{1}{2\sqrt{x}}\sin(x)}$

44. Use the Product Rule (twice) to find the derivative of $x^6 \cdot \cos(x) \cdot \sin(x)$.

$\boxed{x^6(\cos x)^2 - x^6(\sin x)^2 + 6x^5(\cos x)(\sin x)}$ can be simplified to $x^6\cos(2x) + 3x^5\sin(2x)$.

45. If $f'(10) = 8$ and $g'(10) = 9$, is it possible to know the derivative of the function $f(x) + g(x)$ at $x = 10$? If so, what is this number? $\boxed{\text{Yes, 17}}$

46. If $f'(10) = 8$ and $g'(10) = 9$, is it possible to know the derivative of the function $f(x) \cdot g(x)$ at $x = 10$? If so, what is this number? $\boxed{\text{No!}}$

47. True or false?

(a) $(f + g)' = f' + g'$ $\boxed{\text{true}}$

(b) $(f \cdot g)' = f' \cdot g'$ $\boxed{\text{false}}$ A correct right-hand side could be parts (c) or (d).

(c) $(f \cdot g)' = f'g + fg'$ $\boxed{\text{true}}$

(d) $\frac{\mathrm{d}}{\mathrm{d}x}(fg) = f\frac{\mathrm{d}g}{\mathrm{d}x} + g\frac{\mathrm{d}f}{\mathrm{d}x}$ $\boxed{\text{true}}$

(e) $(f \cdot g)' = g'f' + gf'$ $\boxed{\text{false}}$ A correct right-hand side could be $g'f + gf'$ without the extra $'$ in the first term.

(f) $(f/g)' = gf' - fg'$ $\boxed{\text{false}}$ A correct right-hand side could be $(gf' - fg')/g^2$.

48. For $f(x) = 12x^3 + 6x^2$,

(a) At what x value(s) does $f(x) = 0$? $\boxed{x = -\frac{1}{2} \text{ and } x = 0}$

(b) At what x value(s) does $f(x)$ change sign? That is, list values r where either $f(x) < 0$ when x is slightly less than r and $f(x) > 0$ when x is slightly more than r, or $f(x) > 0$ when x is slightly less than r and $f(x) < 0$ when x is slightly more than r. $\boxed{x = -\frac{1}{2} \text{ only}}$

(c) At what x value(s) does $f'(x) = 0$? $\boxed{x = -\frac{1}{3} \text{ and } x = 0}$

(d) At what x value(s) does $f'(x)$ change sign? $\boxed{x = -\frac{1}{3} \text{ and } x = 0}$