Analysis 1, Summer 2024

List 2

Tangent lines, monotonicity, critical points

49. Give the slope of the tangent line to $y = 3x + \dfrac{7}{x}$ at $x = 2$.

    If $f(x) = 3x + \dfrac{7}{x}$, then $f'(x) = 3 - \dfrac{7}{x^2}$ and $f'(2) = 3 - \dfrac{7}{4} = \boxed{\dfrac{5}{4}}$.

50. Give an equation for the tangent line to $y = 3x + \dfrac{7}{x}$ at $x = 2$.

    Slope $\frac{5}{4}$. Since $f(2) = 6 + \frac{7}{2} = \frac{19}{2}$, the point $(2, \frac{19}{2})$ is on this line. An equation for the line through $(2, \frac{19}{2})$ with slope $\frac{5}{4}$ is $\boxed{y = \frac{19}{2} + \frac{5}{4}(x - 2)}$, or $y = \frac{5}{4}x + 7$.

51. Give an equation for the tangent line to $y = \sin(x)$ at $x = \frac{\pi}{3}$. $\boxed{y = \dfrac{\sqrt{3}}{2} + \dfrac{1}{2}\left(x - \dfrac{\pi}{3}\right)}$

☆52. Find a number k so that the tangent line to $y = x^2 + 4x$ at $x = k$ and the tangent line to $y = \frac{1}{5}x^5 - 8x + 1$ at $x = k$ are parallel. $\boxed{k = 2}$

53. Use the fact that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\sin(\tfrac{1}{x})\right] = \frac{-\cos(\frac{1}{x})}{x^2}$$

    to find an equation for the tangent line to $y = \sin(\frac{1}{x})$ at $x = \frac{1}{\pi}$. $\boxed{y = \pi^2 x - \pi}$

54. (a) For what value(s) of x does $x^3 - 18x^2 = 0$? $\boxed{x = 0, x = 18}$

    (b) For what value(s) of x does $3x^2 - 36x = 0$? $\boxed{x = 0, x = 12}$

    (c) For what value(s) of x does $6x - 36 = 0$? $\boxed{x = 6}$

55. At what values of x is the tangent line to $y = x^3 - 18x^2$ horizontal?

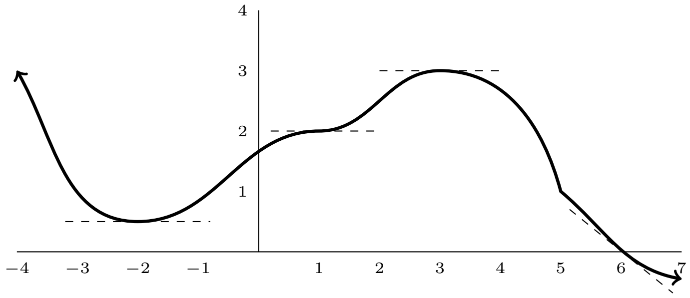
    This is the same as Task 54(b). $\boxed{x = 0, x = 12}$

> A number c in the domain of $f(x)$ is a **critical point** of $f(x)$ if either $f'(c) = 0$ or $f'(c)$ does not exist.
>
> If $f'(a) > 0$ then f is **increasing** at $x = a$.
> If $f'(a) < 0$ then f is **decreasing** at $x = a$.

56. What are the critical points of $x^3 - 18x^2$? $\boxed{x = 0, x = 12}$

57. Find all the critical points of $8x^5 - 57x^4 - 24x^3 + 9$. $\boxed{0, 6, \frac{-3}{10}}$

58. List all the critical points of the function graphed below (portions of its tangent lines at $x = -2$, $x = 1$, $x = 3$, and $x = 6$ are shown as dashed lines).



Critical points are $\boxed{-2, 1, 3, 5}$ (but not $x = 6$).

59. Is the function

$$f(x) = x^8 - 6x^3 + 29x - 12$$

increasing, decreasing, or neither when $x = -1$? $\boxed{\text{increasing}}$

60. (a) On what (possibly infinite) interval or intervals is $2x^3 - 3x^2 - 12x$ increasing?

$\boxed{x < -1 \text{ or } x > 2}$, which is $\boxed{(-\infty, -1) \cup (2, \infty)}$ in interval notation.

(b) On what (possibly infinite) interval or intervals is $2x^3 - 3x^2 - 12x$ decreasing?

$\boxed{-1 < x < 2}$, which is $\boxed{(-1, 2)}$ in interval notation.

61. Suppose $f(x)$ is a function that is increasing when $x = 5$.

(a) Is it possible to know the sign of $f(5)$? (That is, it is possible to know which of $f(5) > 0$ or $f(5) = 0$ or $f(5) < 0$ is true?) $\boxed{\text{No}}$

(b) Is it possible to know the sign of $f'(5)$? $\boxed{\text{Yes: } f'(5) > 0}$

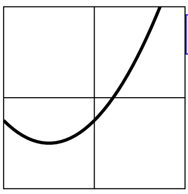
(c) Is it possible to know the sign of $f''(5)$? $\boxed{\text{No}}$

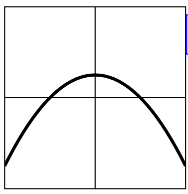
62. On what interval(s) is $x^2 - 8\sqrt{x} + 7$ decreasing? $\boxed{[0, \sqrt[4]{3})}$

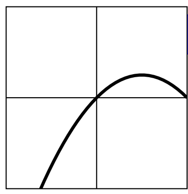
63. List all critical points of $f(x) = \frac{3}{4}x^4 - 7x^3 + 15x^2$ in the interval $[-3, 3]$.

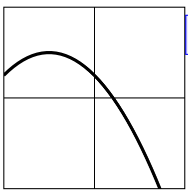
$f'(c) = 0$ for $c = 0, 2, 5$, but only $\boxed{0 \text{ and } 2}$ are in the interval $[-3, 3]$.

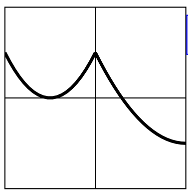
64. For each graph below, is there a critical point at $x = 0$?

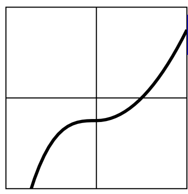
(a)  $\boxed{\text{No}}$

(b)  $\boxed{\text{Yes}}$

(c)  $\boxed{\text{No}}$

(d)  $\boxed{\text{No}}$

(e)  $\boxed{\text{Yes}}$

(f)  $\boxed{\text{Yes}}$

65. The derivative of

$$f(x) = \frac{4x+1}{3x^2-12} \qquad \text{is} \qquad f'(x) = \frac{-4x^2-2x-16}{3x^4-24x^2+48}.$$

Using this, find all the critical points of $f(x)$.

$-4x^2 - 2x - 16 = 0$ has no real solutions, but $3x^4 - 24x^2 + 48 = 0$ when $\boxed{x = 2,\ x = -2}$, so f' does not exist at those points.

66. Find all the critical points of

(a) $f(x) = x^2 - \cos(x)$. $f' = 2x + \sin(x) = 0$ means $\sin(x) = -2x$, which is true only for $\boxed{x = 0}$.

(b) $f(x) = 2x + \cos(x)$. $f' = 2 - \sin(x) = 0$ when $\sin(x) = 2$, but this never happens for real values of x. So this function has $\boxed{\text{no critical points}}$.

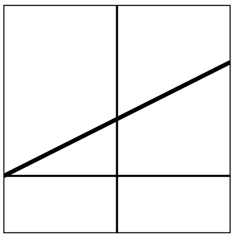
(c) $f(x) = x + 2\cos(x)$. $f' = 1 - 2\sin(x) = 0$ means $\sin(x) = \frac{1}{2}$, which is true for $\boxed{x = \frac{1}{6}\pi + 2k\pi \text{ and } x = \frac{5}{6}\pi + 2k\pi}$, where k can be any integer.

(d) $f(x) = x^2 + x - \sin(x)$. $f' = 2x + 1 - \cos(x) = 0$ when the curves $y = \cos(x)$ and $y = 2x + 1$ intersect. This happens only at $\boxed{x = 0}$.

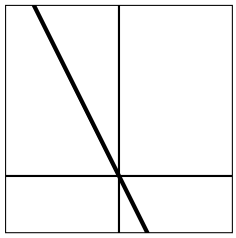
☆ (e) $f(x) = x^2 + x + \cos(x)$. $f' = 2x + 1 - \sin(x) = 0$ when the curves $y = \sin(x)$ and $y = 2x+1$ intersect. There is one point where this occurs, but there is no nice (technically, “closed form”) formula for this value. It is approximately $\boxed{x \approx -0.335418}$.

67. Match the functions (a)-(f) to their derivatives (I)-(VI).

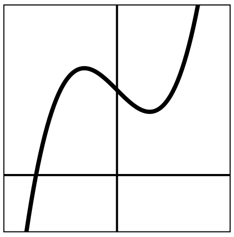
(a)



(I)



(b)



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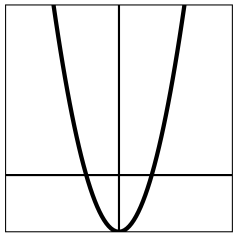
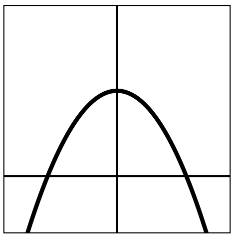


(c)



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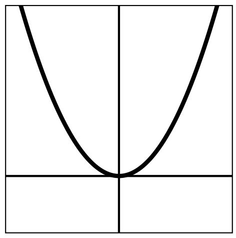
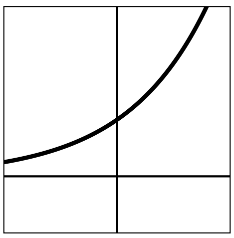


(d)



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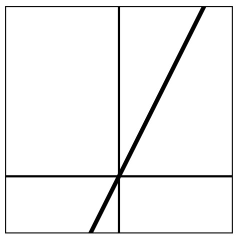
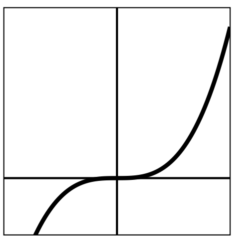
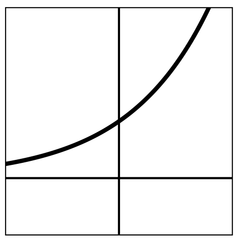


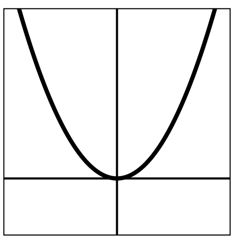
(e)



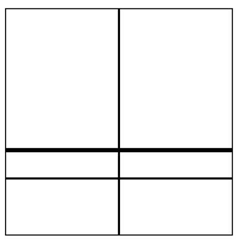
(V)



(f)



(VI)



a-VI, b-II, c-I, d-V, e-III, f-IV