Analysis 1, Summer 2024

List 3

Local extremes, concavity, infection points

68. (a) Calculate the derivative of $5x^2 - 3\sin(x)$. $\boxed{10x - 3\cos(x)}$

(b) Calculate the derivative of $10x - 3\cos(x)$. $\boxed{10 + 3\sin(x)}$

(c) Calculate the derivative of $10 + 3\sin(x)$. $\boxed{3\cos(x)}$

(d) Calculate the derivative of $3\cos(x)$. $\boxed{-3\sin(x)}$

(e) Calculate the derivative of $-3\sin(x)$. $\boxed{-3\cos(x)}$

The **second derivative** of a function is the derivative of its derivative. The second derivative of $y = f(x)$ with respect to x can be written as any of

$$f''(x), \qquad f'', \qquad (f')', \qquad f^{(2)}, \qquad y'', \qquad \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{d}f}{\mathrm{d}x}\right], \qquad \frac{\mathrm{d}^2 f}{\mathrm{d}x^2}, \qquad \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}.$$

We say f is **twice-differentiable** if f'' exists on the entire domain of f. Higher derivatives (third, fourth, etc.) are defined and written similarly.

A twice-differentiable function $f(x)$ is **concave up** at $x = a$ if $f''(a) > 0$.
A twice-differentiable function $f(x)$ is **concave down** at $x = a$ if $f''(a) < 0$.

An **inflection point** is a point where the concavity of a function changes.

69. Compute the following second derivatives:

(a) $f''(x)$ for $f(x) = x^{12}$ $\boxed{132x^{10}}$

(b) $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}$ for $f(x) = x^3 + x^8$ $\boxed{56x^6 + 6x}$

(c) $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ for $y = 8x - 4$ $\boxed{0}$

(d) $\frac{\mathrm{d}^2}{\mathrm{d}x^2}(5x^2 - 7x + 28)$ $\boxed{10}$

(e) $f''(x)$ for $f(x) = -2x^8 + x^6 - x^3$ $\boxed{-112x^6 + 30x^4 - 6x}$

(f) $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}$ for $f(x) = ax^2 + bx + c$ $\boxed{2a}$

70. Find $f'''(x) = \frac{\mathrm{d}^3 f}{\mathrm{d}x^3} = f^{(3)}(x)$ (the third derivative) for $f(x) = x^7$. $\boxed{210x^4}$

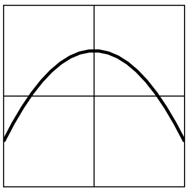
71. Give $f^{(5)}(x) = \frac{\mathrm{d}^5 f}{\mathrm{d}x^5}$ (the fifth derivative) for $f(x) = 5x^2 - 3\sin(x)$.
This is **Task 68(e)**. Answer: $\boxed{-3\cos(x)}$.

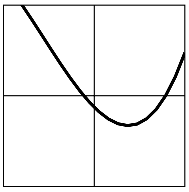
72. (a) Is the function $3x^2 + 8\cos(x)$ concave up or concave down at $x = 0$? $\boxed{\text{concave down}}$

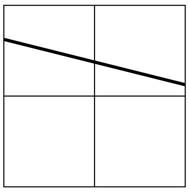
(b) Is the function $3x^2 + 5\cos(x)$ concave up or concave down at $x = 0$? $\boxed{\text{concave up}}$

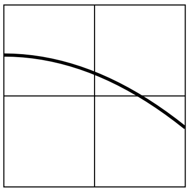
73. On what interval(s) is $54x^2 - x^4$ concave up? $\boxed{-3 < x < 3}$

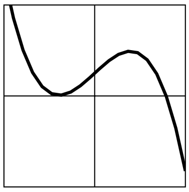
74. For each of the following functions, is $f''(0)$ is positive, zero, or negative?

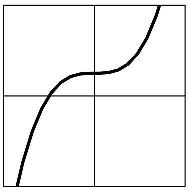
(a)  $\boxed{-}$

(b)  $\boxed{+}$

(c)  $\boxed{0}$

(d)  $\boxed{-}$

(e)  $\boxed{0}$

(f)  $\boxed{0}$

75. For $f(x) = x^3 - x^2 - x$,

(a) At what x value(s) does $f(x)$ change sign? That is, list values r where either $f(x) < 0$ when x is slightly less than r and $f(x) > 0$ when x is slightly more than r, or $f(x) > 0$ when x is slightly less than r and $f(x) < 0$ when x is slightly more than r.

$$\boxed{x = \frac{1-\sqrt{5}}{2}, x = 0, x = \tfrac{1+\sqrt{5}}{2}}$$

(b) At what x value(s) does $f'(x)$ change sign? $\boxed{x = \frac{-1}{3}, x = 1}$

(c) At what x value(s) does $f''(x)$ change sign? $\boxed{x = \frac{1}{3}}$

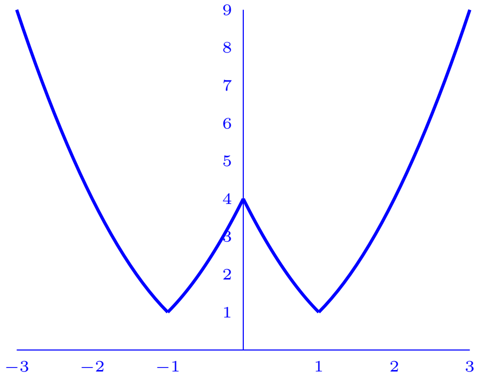
(d) List all inflection points of $x^3 - x^2 - x$. same as (c): $\boxed{x = \frac{1}{3}}$

☆76. Give an example of a function with one local maximum and two local minimums but no inflection points.

In order to avoid inflection points, the function must have the same concavity everywhere. An example that is concave up everywhere is

$$f(x) = \begin{cases} x^2 & \text{if } x < -1 \\ (x+2)^2 & \text{if } -1 \leq x < 0 \\ (x-2)^2 & \text{if } 0 \leq x < 1 \\ x^2 & \text{if } x \geq 1. \end{cases}$$

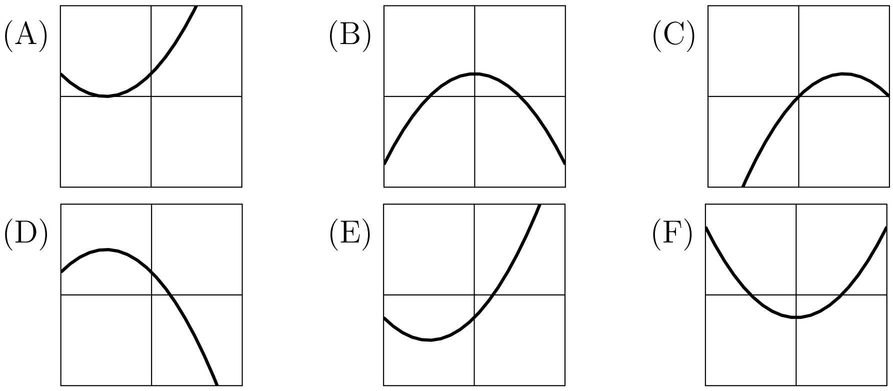
The graph of this is



A similar-looking example that is neither concave up nor concave down is

$$g(x) = |x+1| + |x-1| - |x|.$$

77. Which graph below has $f'(0) = 1$ and $f''(0) = -1$? $\boxed{\text{C}}$



For a twice-differentiable function $f(x)$ with a critical point at $x = c$, ...

The Second Derivative Test:
- If $f''(c) > 0$ then f has a local minimum at $x = c$.
- If $f''(c) < 0$ then f has a local maximum at $x = c$.
- If $f''(c) = 0$ the test is inconclusive.

The First Derivative Test:
- If $f'(x) < 0$ to the left of $x = c$ and $f'(x) > 0$ to the right of $x = c$ then f has a local minimum at $x = c$.
- If $f'(x) > 0$ to the left of $x = c$ and $f'(x) < 0$ to the right of $x = c$ then f has a local maxium at $x = c$.
- If $f'(x)$ has the same sign on both sides of $x = c$ then $x = c$ is neither a local minimum nor a local maximum.

78. Find all critical points of

$$4x^3 + 21x^2 - 24x + 19$$

and classify each as a local minimum, local maximum, or neither.

$\boxed{\text{local max at } x = -4\text{, local min at } x = 1/2}$

79. Find and classify the critical points of $f(x) = x^4 - 4x^3 - 36x^2 + 18$.

$\boxed{x = -3 \text{ is local min. } x = 0 \text{ is local max. } x = 6 \text{ is local min.}}$

80. Find the inflection points of the function from Task 79. $\boxed{1 - \sqrt{7},\ 1 + \sqrt{7}}$

☆81. Find and classify the critical points of $f(x) = x(6-x)^{2/3}$.

After simplifying, $f'(x) = \dfrac{18 - 5x}{3(6-x)^{1/3}}$, so the critical points are $x = \frac{18}{5}$ (where f' is zero) and $x = 6$ (where f' doesn't exist). The fact that $\boxed{x = \frac{18}{5} \text{ is a local max}}$ can be found from the First or the Second Derivative Test, but the fact that $\boxed{x = 6 \text{ is a local min}}$ requires the First D. Test because $f''(6)$ is not defined.

82. Find and classify the critical points of $\frac{3}{2}x^4 - 16x^3 + 63x^2 - 108x + 51$.

$\boxed{x = 2 \text{ is local min. } x = 3 \text{ is } \textit{neither.}}$

83. Label each of following statements as "true" or "false":
   (a) Every critical point of a differentiable function is also a local minimum. $\boxed{\text{false}}$
   (b) Every local minimum of a differentiable function is also a critical point. $\boxed{\text{true}}$
   (c) Every critical point of a differentiable function is also an inflection point. $\boxed{\text{false}}$
   (d) Every inflection point of a differentiable function is also a critical point. $\boxed{\text{false}}$

84. A twice-differentiable function $f(x)$ has the following properties:

$$f(4) = 2 \qquad f'(4) = 18 \qquad f''(4) = 0,$$
$$f(7) = 19 \qquad f'(7) = 0 \qquad f''(7) = -1.$$

Label each of following statements as "true", "false", or "cannot be determined":
   (a) f has a critical point at $x = 4$. $\boxed{\text{false}}$
   (b) f has a local maximum at $x = 4$. $\boxed{\text{false}}$
   (c) f has an absolute maximum at $x = 4$. $\boxed{\text{false}}$
   (d) f has an inflection point at $x = 4$. $\boxed{\text{cannot be determined}}$
   (e) f has a critical point at $x = 7$. $\boxed{\text{true}}$
   (f) f has a local maximum at $x = 7$. $\boxed{\text{true}}$
   (g) f has an absolute maximum at $x = 7$. $\boxed{\text{cannot be determined}}$
   (h) f has an inflection point at $x = 7$. $\boxed{\text{false}}$

☆85. What is the maximum number of inflection points that a function of the form

$$\_\_x^6 + \_\_x^5 + \_\_x^4 + \_\_x^3 + \_\_x^2 + \_\_x + \_\_$$

can have? $\boxed{4}$ because f'' will be a degree-4 polynomial.

Basic functions: $\frac{\mathrm{d}}{\mathrm{d}x}\big[x^p\big] = px^{p-1}$, $\frac{\mathrm{d}}{\mathrm{d}x}\big[\sin(x)\big] = \cos(x)$, $\frac{\mathrm{d}}{\mathrm{d}x}\big[\cos(x)\big] = -\sin(x)$.

Sum Rule: $(f+g)' = f' + g'$ **Product Rule:** $(f \cdot g)' = fg' + f'g$

Chain Rule: $(f(g))' = f'(g) \cdot g'$ **Quotient Rule:** $(f/g)' = \dfrac{gf' - fg'}{g^2}$

86. Give an equation for the tangent line to $y = \sin(\pi x)$ at $x = 2$.

$y(2) = \sin(2\pi) = 0$. Slope $y'(2) = \pi\cos(2\pi) = \pi$. Line: $\boxed{y = \pi(x-2)}$.

87. Find the derivative of $\sin(\sqrt{\cos(2x^3+8)})$.

$$\boxed{\cos(\sqrt{\cos(2x^3+8)}) \cdot \frac{1}{2\sqrt{\cos(2x^3+8)}} \cdot \big(-\sin(2x^3+8)\big) \cdot 6x^2}$$

88. (a) Use the Quotient Rule to differentiate $\dfrac{\sin(x)}{x^4}$. $\boxed{\dfrac{x^4\cos(x) - \sin(x)(4x^3)}{x^8}}$

(b) Use the Product Rule to differentiate $x^{-4}\sin(x)$. $\boxed{x^{-4}\cos(x) + (-4x^{-5})\sin(x)}$

(c) Use algebra to compare your answers from parts (a) and (b). $\boxed{\text{they are equal}}$

89. At $x = 2$, is $\dfrac{x^2}{1+x^3}$ increasing, decreasing, or neither?

$\boxed{\text{decreasing}}$ because $f'(2) = \frac{-4}{27} < 0$.

90. At $x = 0$, is $\sqrt{2+\sin(x)}$ concave up, concave down, or neither?

$\boxed{\text{concave down}}$ because $f''(0) = \frac{-1}{8\sqrt{2}} < 0$.

91. Match the functions (a)-(d) with their derivatives (I)-(IV).

(a) $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$ (I) $\sec(x)\tan(x) = \dfrac{\sin(x)}{(\cos(x))^2}$

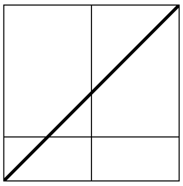
(b) $\cot(x) = \dfrac{\cos(x)}{\sin(x)}$ (II) $-(\csc(x))^2 = \dfrac{-1}{(\sin(x))^2}$

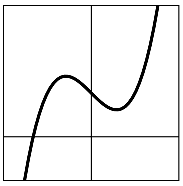
(c) $\sec(x) = \dfrac{1}{\cos(x)}$ (III) $(\sec(x))^2 = \dfrac{1}{(\cos(x))^2}$

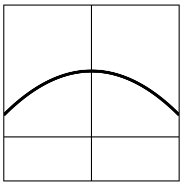
(d) $\csc(x) = \dfrac{1}{\sin(x)}$ (IV) $-\csc(x)\cot(x) = \dfrac{-\cos(x)}{(\sin(x))^2}$

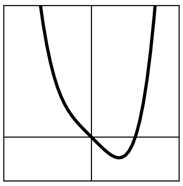
$\boxed{\text{(a)-(III), (b)-(II), (c)-(I), (d)-(IV)}}$

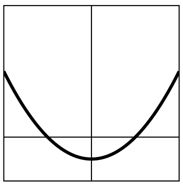
92. Match the functions (a)-(g) to their <u>*second*</u> derivatives (I)-(VII).

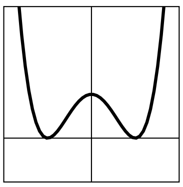
(a) 

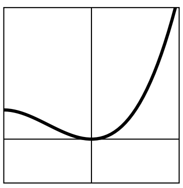
(b) 

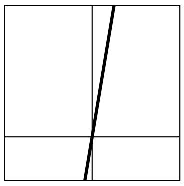
(c) 

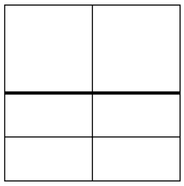
(d) 

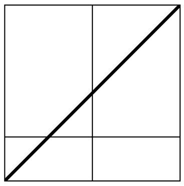
(e) 

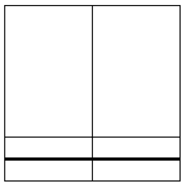
(f) 

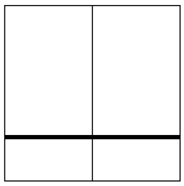
(g) 

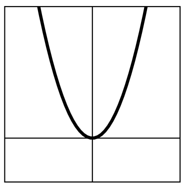
(I) 

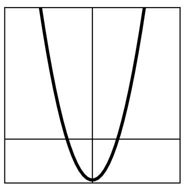
(II) 

(III) 

(IV) 

(V) 

(VI) 

(VII) 

(a)-(V)
(b)-(I)
(c)-(IV)
(d)-(VI)
(e)-(II)
(f)-(VII)
(g)-(III)