Analysis 1, Summer 2024

List 4

Limits

93. Calculate the following limits:

(a) $\lim\limits_{x\to 7} \dfrac{x^2-4x-21}{x^2-11x+28} = \boxed{\dfrac{10}{3}}$

(b) $\lim\limits_{x\to 0} \dfrac{x^3-8x^2+3x+5}{x^9-6x^5+x^4-12x+1} = \boxed{5}$

(c) $\lim\limits_{x\to \infty} \dfrac{3x^3-2x+1}{6x^3+x^2+x+19} = \boxed{\dfrac{1}{2}}$

(d) $\lim\limits_{x\to \infty} \dfrac{3x^2-2x+1}{6x^3+x^2+x+19} = \boxed{0}$

(e) $\lim\limits_{x\to 0} \left(\dfrac{8x-1}{x-x^2} + \dfrac{1}{x}\right) = \boxed{7}$

(f) $\lim\limits_{n\to \infty} \left(\sqrt{9n^2+5n} - 3n\right) = \boxed{\dfrac{5}{6}}$

(g) $\lim\limits_{x\to \infty} \left(\sqrt{9x^2+5x} - 3x\right) = \boxed{\dfrac{5}{6}}$

(h) $\lim\limits_{x\to \infty} (4^x+1)^{1/4} = \boxed{\infty}$

☆ (i) $\lim\limits_{x\to \infty} (4^x+x)^{1/x} = \boxed{4}$

94. At $x=1$, does the function

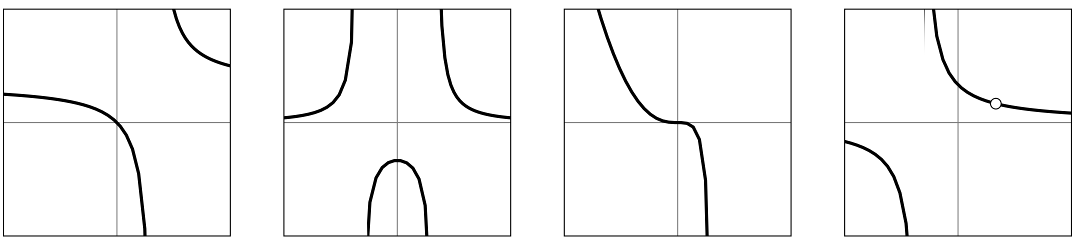
$$f(x) = \begin{cases} 2x-1 & \text{if } x \le 1, \\ x/3 & \text{if } 1 < x < 9, \\ \sqrt{x} & \text{if } x \ge 9 \end{cases}$$

have a jump, hole, vertical asymptote, or none of these? $\boxed{\text{Jump}}$
At $x=9$? $\boxed{\text{None}}$

95. Match the functions with their graphs:

(a) $\dfrac{1}{x^2-1}$ $\boxed{\text{(II)}}$ (c) $\dfrac{x-1}{x^2-1}$ $\boxed{\text{(IV)}}$ (b) $\dfrac{x^3}{x-1}$ $\boxed{\text{(III)}}$ (d) $\dfrac{x}{x-1}$ $\boxed{\text{(I)}}$

(I) (II) (III) (IV)



96. Suppose $\lim\limits_{x\to 10^-} f(x) = 2$.

(a) If the graph of f has a hole at $x=10$, is it possible to know the value of $\lim\limits_{x\to 10^+} f(x)$ from only this information? $\boxed{\text{Yes: 2}}$

(b) If the graph of f has a hole at $x=10$, is it possible to know the value of $f(2)$ from only this information? $\boxed{\text{No}}$

(c) If the graph of f has a jump at $x=10$, is it possible to know the value of $\lim\limits_{x\to 10^+} f(x)$ from only this information? $\boxed{\text{No}}$

(d) If the graph of f has a vertical asymptote at $x = 10$, is it possible to know the value of $\lim_{x\to 10^+} f(x)$ from only this information?

$\boxed{\text{No}}$, but it must be either $+\infty$ or $-\infty$.

(e) If the graph of f has a vertical asymptote at $x = 10$, is it possible to know the value of $\lim_{x\to 10^+} |f(x)|$ from only this information? $\boxed{\text{Yes: } \infty}$

$\boxed{\text{Jump}}$ because $\lim_{x\to 9^-} f(x) = \log_3(9) = 2$ does not equal $\lim_{x\to 9^+} f(x) = \sqrt{9} = 3$.

97. Find the following limits, if they exist:

(a) $\lim_{x\to 0^+} \frac{|x|}{x} = \boxed{1}$ (b) $\lim_{x\to 0^-} \frac{|x|}{x} = \boxed{-1}$ (c) $\lim_{x\to 0} \frac{|x|}{x}$ $\boxed{\text{does not exist}}$

98. For which value(s) of the parameter a does the function

$$f(x) = \frac{x^2 - a}{x^2 + a}$$

have a vertical asymptote at $x = 3$? $\boxed{a = -9}$

99. For which value(s) of the parameter a is the function from Task 98 continuous?
any $\boxed{a > 0}$

100. For which value(s) of the parameter a is the function

$$f(x) = \begin{cases} x - 6a & \text{if } x < a \\ ax + 4 & \text{if } x \geq a \end{cases}$$

continuous? $\boxed{a = -4,\ a = -1}$

101. Which limit expression below gives the derivative of x^3 at the point $x = 2$? $\boxed{\text{(C)}}$

(A) $\lim_{x\to 2} \frac{x^3 - 8}{x}$ (C) $\lim_{h\to 0} \frac{(2+h)^3 - 8}{h}$

(B) $\lim_{h\to 0} \frac{h^3 - 8}{h}$ (D) $\lim_{h\to 0} \frac{(2+h)^3 - h^3}{h}$

L'Hôpital's Rule: if $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$ and $\lim_{x\to a} \frac{f(x)}{g(x)}$ exists, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

The same substitution works if $\lim_{x\to a} f(x) = \infty$ or $-\infty$ and $\lim_{x\to a} g(x) = \infty$ or $-\infty$.
And also for one-sided limits and for $x \to \infty$ and $x \to -\infty$.

102. Calculate $\lim_{x\to 1} \frac{3x^3 + 4x^2 - 13x + 6}{2x^4 + x^3 - x^2 + x - 3}$.

$$\lim_{x\to 1} \frac{9x^2 + 8x - 13}{8x^3 + 3x^2 - 2x + 1} = \frac{9 + 8 - 13}{8 + 3 - 2 + 1} = \frac{4}{10} = \boxed{\frac{2}{5}}$$

103. (a) Find $\lim_{x\to 1} \frac{x^2-4x+18}{3x^2+10}$. $\frac{1-4+18}{3+10} = \boxed{\frac{15}{13}}$

(b) Find $\lim_{x\to 1} \frac{2x-4}{6x}$. $\frac{2(1)-4}{6(1)} = \boxed{\frac{-1}{3}}$

(c) Why are the answers to (a) and (b) not equal?
Because $\frac{x^2-4x+18}{3x^2+10}$ is NOT $\frac{0}{0}$ or $\frac{\infty}{\infty}$ when $x = 1$.

104. Find $\lim_{x\to 0} \frac{2\sin(x)-\sin(2x)}{x-\sin(x)}$.

$$\begin{aligned}\lim_{x\to 0} \frac{2\sin(x)-\sin(2x)}{x-\sin(x)} &= \lim_{x\to 0} \frac{2\cos(x)-2\cos(2x)}{1-\cos(x)} \\ &= \lim_{x\to 0} \frac{-2\sin(x)+4\sin(2x)}{\sin(x)} \\ &= \lim_{x\to 0} \frac{-2\cos(x)+8\cos(2x)}{\cos(x)} \\ &= \frac{-2+8}{1} = \boxed{6}\end{aligned}$$

$$(e^x)' = e^x \qquad \big(\ln(x)\big)' = \frac{1}{x}$$

105. For the function $f(x) = x^2e^{-x}$, find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to 0} f(x)$ and $\lim_{x\to -\infty} f(x)$.

(a) $\lim_{x\to\infty} x^2e^{-x} = \lim_{x\to\infty} \frac{x^2}{e^x} \overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{2x}{e^x} \overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{2}{e^x} = \boxed{0}$.

(b) Do not use L'Hôpital! Just plug in $x = 0$ to get $\frac{0^2}{e^{-0}} = \frac{0}{e^0} = \frac{0}{1} = \boxed{0}$.

(c) $\lim_{x\to -\infty} x^2e^{-x} = \lim_{x\to\infty} x^2e^x = \boxed{+\infty}$.

106. Calculate $\lim_{x\to 4} \frac{\sin(\pi x)}{\ln(x-3)} = \lim_{x\to 4} \frac{\pi\cos(\pi x)}{1/(x-3)} = \frac{\pi\cos(4\pi)}{1/1} = \boxed{\pi}$

107. Give an equation for the tangent line to $y = e^{3x}(\cos(4x))^5$ at $x = 0$. $\boxed{y = 3x+1}$

108. Is $y = e^{\sin(x)}$ concave up or concave down when $x = \pi$? $\boxed{\text{concave down}}$ because $f''(\pi) = -1$ is negative.

109. Find the local extremes of $f(x) = \sqrt{x} - \ln(x)$. $\boxed{x = 4 \text{ is a local min}}$

110. Find the limit $\lim_{x\to 0^+} \frac{\ln(x)}{1/x}$. L'Hôpital: $\lim_{x\to 0^+} \frac{\ln(x)}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2}$.
Then $\frac{1/x}{-1/x^2} = \frac{-x^2}{x} = -x$, so the limit is $-0 = \boxed{0}$.

111. Find the limit $\lim_{x \to 0^+} x \ln(x)$ Algebra: $x \ln(x) = \dfrac{\ln(x)}{1/x}$, so this is the same as the previous task! $\boxed{0}$

112. (a) Calculate $\lim_{n \to \infty} n \cdot \ln\left(1 + \frac{1}{n}\right)$ using algebra and L'Hôpital. $\boxed{1}$

(b) Calculate $\lim_{n \to \infty} \ln\left(\left(1 + \frac{1}{n}\right)^n\right)$ using log rules and L'Hôpital. $\boxed{1}$

(c) Calculate $\lim_{n \to \infty} \left(1 + \dfrac{1}{n}\right)^n$ using the fact that $x = e^{\ln(x)}$ and therefore

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e^{\ln\left(\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n\right)}.$$

$e^1 = \boxed{e}$