Analysis 1, Summer 2024

List 5

Definite and indefinite integrals

> An **anti-derivative** of $f(x)$ is a function whose derivative is $f(x)$.
> In symbols, $F(x)$ is an anti-derivative of $f(x)$ if $F'(x) = f(x)$.

113. (a) Give an anti-derivative of $10x^9$.
That is, give a function $F(x)$ for which $F'(x) = 10x^9$.

(b) Give another anti-derivative of $10x^9$.

(c) Give another anti-derivative of $10x^9$.

(d) Give another anti-derivative of $10x^9$.

All answers will be of the form $\boxed{x^{10} + C}$. These might include x^{10} or $x^{10} + 1$ or $x^{10} - 12345$, etc.

114. Give an anti-derivative of $\sin(x)$. $\boxed{-\cos(x)}$ or any $-\cos(x) + C$

115. Give an anti-derivative for each of the following functions:

(a) x^3 $\boxed{\frac{1}{4}x^4}$ (e) $-3x^{15}$ $\boxed{\frac{-3}{16}x^{16}}$ (i) $\frac{-4}{3}x^7$ $\boxed{\frac{-1}{6}x^8}$

(b) $12x^5$ $\boxed{2x^6}$ (f) $\frac{1}{2}x^2$ $\boxed{\frac{1}{6}x^3}$ (j) $5\sin(x)$ $\boxed{-5\cos(x)}$

(c) $12x^4$ $\boxed{\frac{12}{5}x^5}$ (g) x^{5000} $\boxed{\frac{1}{5001}x^{5001}}$ (k) $2\cos(x)$ $\boxed{2\sin(x)}$

(d) x^{15} $\boxed{\frac{1}{16}x^{16}}$ (h) $\frac{3}{5}x^{12}$ $\boxed{\frac{3}{65}x^{13}}$ (ℓ) e^x $\boxed{e^x}$

> An **indefinite integral** describes all the anti-derivatives of a function. We write
> $$\int f(x)\,dx = F(x) + C,$$
> where $F(x)$ is any function for which $F'(x) = f(x)$.

116. Find $\int (2x^5 + 3x - 9)\,dx = \boxed{\frac{1}{3}x^6 + \frac{3}{2}x^2 - 9x + C}$

Find $\int (2u^5 + 3u - 9)\,du = \boxed{\frac{1}{3}u^6 + \frac{3}{2}u^2 - 9u + C}$

118. Give each of the following indefinite integrals using basic derivative knowledge:

(a) $\int x^{372.5}\,dx = \boxed{\frac{1}{373.5}x^{373.5} + C}$

(b) $\int \frac{1}{x}\,dx = \boxed{\ln(x) + C}$

(c) $\int e^x\,dx = \boxed{e^x + C}$

(d) $\int 97^x \, dx = \boxed{\frac{1}{\ln(97)} 97^x + C}$

(e) $\int -\sin(x) \, dx = \boxed{\cos(x) + C}$

(f) $\int \sin(x) \, dx = \boxed{-\cos(x) + C}$

(g) $\int \cos(x) \, dx = \boxed{\sin(x) + C}$

(h) $\int 5t^9 \, dt = \boxed{\frac{1}{2} t^{10} + C}$

119. If $u = 6x^2 - 5$, give a formula for du (this formula will have x and dx in it) and a formula for dx (this formula will have x and du in it).

$\boxed{du = 12x \, dx}$ and $\boxed{dx = \frac{du}{12x}}$

The notation $\boldsymbol{g(x)}\Big|_{\boldsymbol{x=a}}^{\boldsymbol{x=b}}$ or $g(x)\Big|_a^b$ means to do the subtraction $g(b) - g(a)$.

120. Calculate $\frac{1}{3}x^3\Big|_{x=1}^{x=3}$ $= \frac{1}{3}(3)^3 - \frac{1}{3}(1)^3 = 9 - \frac{1}{3} = \boxed{\frac{26}{3}}$

121. Calculate $(x^3 + \frac{1}{2}x)\Big|_{x=1}^{x=5}$. $\boxed{126}$

Calculate $\frac{1-x}{e^x}\Big|_{x=0}^{x=1}$. $\boxed{-1}$

The **definite integral** $\int_a^b f(x) \, dx$, spoken as "the integral from a to b of $f(x)$ with respect to x", is the (signed) area of the region with $x = a$ on the left, $x = b$ on the right, $y = 0$ at the bottom, and $y = f(x)$ at the top (but if $f(x) < 0$ for some x or if $b < a$ then it's possible for the area to be negative).
The **Fundamental Theorem of Calculus** says that

$$\int_a^b f(x) \, dx \; = \; F(x)\Big|_{x=a}^{x=b} \; = \; F(b) - F(a),$$

where $F(x)$ is any function for which $F'(x) = f(x)$.

123. Calculate $\int_1^3 x^2 \, dx$ using the FTC. This is exactly **Task ??**. Answer: $\boxed{\frac{26}{3}}$

124. Write, in symbols, the integral from zero to six of x^2 with respect to x, then find the value of that definite integral. $\boxed{\int_0^6 x^2 \, dx = 72}$

125. Evaluate (meaning find of the value of) the following definite integrals using common area formulas.

(a) $\int_3^9 2\,\mathrm{d}x = \boxed{12}$

(b) $\int_3^9 -2\,\mathrm{d}x = \boxed{-12}$

(c) $\int_0^5 x\,\mathrm{d}x = \boxed{\dfrac{25}{2}}$

(d) $\int_{-2}^4 |x|\,\mathrm{d}x = \boxed{10}$

(e) $\int_{-2}^4 x\,\mathrm{d}x = \boxed{6}$

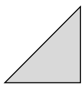
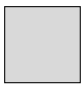
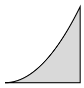
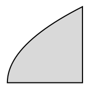
(f) $\int_0^5 3x\,\mathrm{d}x = \boxed{\dfrac{75}{2}}$

(g) $\int_1^5 3x\,\mathrm{d}x = \boxed{36}$

(h) $\int_{-4}^4 \sqrt{16-x^2}\,\mathrm{d}x = \boxed{8\pi}$

(i) $\int_0^7 \sqrt{49-x^2}\,\mathrm{d}x = \boxed{\dfrac{49}{4}\pi}$

126. Match the shapes (a)-(d) with the integral (I)-(IV) that is most likely to calculate its area.

(a)  (b)  (c)  (d)  $\boxed{\text{a-II, b-IV, c-III, d-I}}$

(I) $\int_0^1 \sqrt{x}\,\mathrm{d}x$ (II) $\int_0^1 x\,\mathrm{d}x$ (III) $\int_0^1 x^2\,\mathrm{d}x$ (IV) $\int_0^1 1\,\mathrm{d}x$

127. Find $\int_0^1 \sqrt{x}\,\mathrm{d}x$. $\boxed{\dfrac{2}{3}}$

128. Evaluate the following definite integrals using the FTC. Your answer for each should be a number.

(a) $\int_{-3}^9 2\,\mathrm{d}x = 2x\Big|_{x=-3}^{x=9} = 18-(-6) = \boxed{24}$

(b) $\int_1^5 3x\,\mathrm{d}x = \frac{3}{2}x^2\Big|_{x=1}^{x=5} = \frac{75}{2} - \frac{3}{2} = \boxed{36}$

(c) $\int_1^{12} \dfrac{1}{x}\,\mathrm{d}x = \ln(x)\Big|_{x=1}^{x=12} = \ln(12) - \ln(0) = \boxed{\ln(12)}$

(d) $\int_0^9 (x^3-9x)\,\mathrm{d}x = (\frac{1}{4}x^4 - \frac{9}{2}x^2)\Big|_{x=0}^{x=9} = \frac{5103}{4} - 0 = \boxed{\dfrac{5103}{4}}$

(e) $\int_0^\pi \sin(t)\,\mathrm{d}t = (-\cos(t))\Big|_{t=0}^{t=\pi} = -\cos(\pi) - (-\cos(0)) = \boxed{2}$

(f) $\int_2^8 3 \cdot \sqrt{u}\, \mathrm{d}u = 2x^{3/2}\Big|_{x=2}^{x=8} = 32\sqrt{2} - 4\sqrt{2} = \boxed{28\sqrt{2}}$

(g) $\int_0^1 (e^x + x^e)\, \mathrm{d}x = \boxed{\dfrac{e^2}{1+e}}$

(h) $\int_{-1}^1 x^2\, \mathrm{d}x = \boxed{\dfrac{2}{3}}$

(i) $\int_1^3 t\, \mathrm{d}t = \boxed{4}$

(j) $\int_9^9 \sin(x^2)\, \mathrm{d}x = \boxed{0}$

(k) $\int_0^5 \cos(x)\, \mathrm{d}x = \boxed{\sin(5)}$

129. Evaluate the following definite integrals using the FTC. Your answers will be formulas.

(a) $\int_a^9 2\, \mathrm{d}x = 2x\Big|_{x=a}^{x=9} = \boxed{18 - 2a}$

(b) $\int_1^5 kx\, \mathrm{d}x = \frac{k}{2}x^2\Big|_{x=1}^{x=5} = \frac{25k}{2} - \frac{k}{2} = \boxed{12k}$

(c) $\int_1^t \frac{1}{x}\, \mathrm{d}x = \ln(x)\Big|_{x=1}^{x=t} = \ln(t) - \ln(1) = \boxed{\ln(t)}$

(d) $\int_0^9 (x^p - qx)\, \mathrm{d}x = \boxed{\dfrac{9^{p+1}}{p+1} - \dfrac{81q}{2}}$

130. If $\int_1^4 f(x)\, \mathrm{d}x = 12$ and $\int_1^6 f(x)\, \mathrm{d}x = 15$, what is the value of $I = \int_4^6 f(x)\, \mathrm{d}x$?

$\int_1^6 f(x)\, \mathrm{d}x = \int_1^4 f(x)\, \mathrm{d}x + \int_4^6 f(x)\, \mathrm{d}x$, so $12 + I = 14$, and thus $I = \boxed{3}$.

131. If $\int_0^1 f(x)\, \mathrm{d}x = 7$ and $\int_0^1 g(x)\, \mathrm{d}x = 3$, calculate each of the following or say that there is not enough information to possibly do the calculation.

(a) $\int_0^1 \big(f(x) + g(x)\big)\, \mathrm{d}x = \boxed{10}$

(b) $\int_0^1 \big(f(x) - g(x)\big)\, \mathrm{d}x = \boxed{4}$

(c) $\int_0^1 \big(f(x) \cdot g(x)\big)\, \mathrm{d}x$ $\boxed{\text{not enough info}}$

(d) $\int_0^1 \big(5f(x)\big)\, \mathrm{d}x = \boxed{35}$

(e) $\int_0^1 \big(f(x)^5\big)\, \mathrm{d}x$ $\boxed{\text{not enough info}}$

132. Simplify $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\displaystyle\int_1^t \frac{1}{x}\,\mathrm{d}x\right)$ to a formula that does not include x (assume $t > 1$). $\boxed{\dfrac{1}{t}}$

133. Simplify $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\displaystyle\int_3^t \frac{\sin(x)}{x}\,\mathrm{d}x\right)$ to a formula that does not include x (assume t $\boxed{\dfrac{\sin t}{t}}$

134. Simplify $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\displaystyle\int_0^{t^2} \sin(x)\,\mathrm{d}x\right)$ and $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\displaystyle\int_0^{t^2} \sin(x^2)\,\mathrm{d}x\right)$ to formulas that do not include x. $\boxed{2t\sin(t^2)}$ and $\boxed{2t\sin(t^4)}$

135. Given that $\int \ln(x)\,\mathrm{d}x = x\ln(x) - x + C$, evaluate $\displaystyle\int_1^{e^5} \ln(x)\,\mathrm{d}x$.

$$x\ln(x) - x\Big|_{x=1}^{x=e^5} = e^5\ln(e^5) - e^5 - (\ln(1) - 1) = 5e^5 - e^5 - (-1) = \boxed{4e^5 + 1}$$

Substitution: $\displaystyle\int f(u(x)) \cdot u'(x)\,\mathrm{d}x = \int f(u)\,\mathrm{d}u$

136. (a) Re-write $\displaystyle\int \frac{x}{(6x^2-5)^3}\,\mathrm{d}x$ as $\int \ldots\,\mathrm{d}u$ using the substitution $u = 6x^2 - 5$.

$\mathrm{d}u = 12x\,\mathrm{d}x$, so $x\,\mathrm{d}x = \frac{1}{12}\,\mathrm{d}u$ and the integral is $\boxed{\displaystyle\int \frac{1}{12u^3}\,\mathrm{d}u}$.

(b) Find $\displaystyle\int \frac{x}{(6x^2-5)^3}\,\mathrm{d}x$. (Your final answer should not have u at all.) $\boxed{\dfrac{-1}{24(6x^2-5)^2}}$

137. (a) Re-write $\int x^3\sin(x^4)\,\mathrm{d}x$ as $\int \ldots\,\mathrm{d}u$ using the substitution $u = x^4$. $\boxed{\int \frac{1}{4}\sin(u)\,\mathrm{d}u}$

(b) Find $\int x^3\sin(x^4)\,\mathrm{d}x = \boxed{-\dfrac{1}{4}\cos(x^4) + C}$

138. (a) Re-write $\int x\sin(x^4)\,\mathrm{d}x$ as $\int \ldots\,\mathrm{d}u$ using the substitution $u = x^2$. $\boxed{\int \frac{1}{2}\sin(u^2)\,\mathrm{d}u}$

☆ (b) Find $\int x\sin(x^4)\,\mathrm{d}x$ There is literally no "elementary" formula for this. You might see $\sqrt{\pi/8}S(x^2\sqrt{2/\pi})$ in some sources, but this is just re-writing the integral using a special short-hand for this "Fresnel" integral.

139. Find $\displaystyle\int \frac{x^4 - x^3 - 1}{4x^5 - 5x^4 - 20x + 3}\,\mathrm{d}x$ using substitution. $= \boxed{\frac{1}{20}\ln(4x^5 - 5x^4 - 20x + 3) + C}$

140. Find $\displaystyle\int \cot(x)\,\mathrm{d}x = \int \frac{\cos(x)}{\sin(x)}\,\mathrm{d}x$ using substitution.
With $u = \sin(x)$, $\mathrm{d}u = \cos(x)\,\mathrm{d}x$, so

$$\int \frac{\cos(x)\,\mathrm{d}x}{\sin(x)} = \int \frac{\mathrm{d}u}{u} = \ln|u| + C = \boxed{\ln|\sin(x)| + C}$$

Technically the "C" could actually be a piecewise function that is constant on each interval where $\ln|\sin(x)|$ is continuous. But it is common to just write "$+C$" anyway.

141. Which of the following has the same value as $\displaystyle\int_2^4 \frac{3x^2-2}{\ln(x^3-2x+1)}\,\mathrm{d}x$?

(A) $\displaystyle\int_5^{57}\frac{1}{\ln(u)}\,\mathrm{d}u$ (B) $\displaystyle\int_2^{4}\frac{1}{\ln(u)}\,\mathrm{d}u$ (C) $\displaystyle\int_{10}^{46}\frac{1}{\ln(u)}\,\mathrm{d}u$ (D) $\displaystyle\int_1^{2}\frac{1}{\ln(u)}\,\mathrm{d}u$

Using the substitution $u = x^3-2x+1$, we have $\mathrm{d}u = (3x^2-2)\,\mathrm{d}x$, so

$$\frac{(3x^2-2)\,\mathrm{d}x}{\ln(x^3-2x+1)} = \frac{\mathrm{d}u}{\ln(u)}.$$

The values 2 and 4 are x-values, so **these change** when we write $\displaystyle\int \ldots \mathrm{d}u$.

When $x=2$, $u=2^3-2(2)+1=5$. When $x=4$, $u=4^3-2(4)+1=57$. $\boxed{\text{(A)}}$

142. Find the following integrals using substitution:

(a) $\displaystyle\int (5-x)^{10}\,\mathrm{d}x = \int -u^{10}\,\mathrm{d}u = \tfrac{-1}{11}u^{11}+C = \boxed{\tfrac{-1}{11}(5-x)^{11}+C}$
using $u=5-x$, so $\mathrm{d}u = -\,\mathrm{d}x$.

(b) $\displaystyle\int_1^3 \frac{x}{(6x^2-5)^3}\,\mathrm{d}x$

Indefinite: $\displaystyle\int \frac{x}{(6x^2-5)^3}\,\mathrm{d}x = \int \tfrac{1}{12}u^{-3}\,\mathrm{d}u = \tfrac{-1}{24}u^{-2}+C = \frac{-1}{24(6x^2-5)^2}+C$
using $u=6x^2-5$, so $\mathrm{d}u = 12x\,\mathrm{d}x$. Then the definite integral is

$$\int_1^3 \frac{x}{(6x^2-5)^3}\,\mathrm{d}x = \left.\frac{-1}{24(6x^2-5)^2}\right|_{x=1}^{x=3} = \boxed{\frac{100}{2401}}.$$

Alternatively, when $x=1, u=6(1)^2-5=1$ and when $x=3, u=6(3)^2-5=49$, so this is

$$\int_1^{49}\tfrac{1}{12}u^{-3}\,\mathrm{d}u = \left.\frac{-1}{24u^2}\right|_{u=1}^{u=49} = \boxed{\frac{100}{2401}}.$$

(c) $\displaystyle\int \sqrt{4x+3}\,\mathrm{d}x = \int \tfrac14 u^{1/2}\,\mathrm{d}u = \tfrac16 u^{3/2}+C = \boxed{\tfrac16(4x+3)^{3/2}+C}$
using $u=4x+3$, so $\mathrm{d}u = 4\,\mathrm{d}x$.

(d) $\displaystyle\int_0^{\sqrt{\pi}} x\sin(x^2)\,\mathrm{d}x$

Indefinite: $\displaystyle\int x\sin(x^2)\,\mathrm{d}x = \int \tfrac12\sin(u)\,\mathrm{d}u = \tfrac{-1}{2}\cos(u)+C = \tfrac{-1}{2}\cos(x^2)+C$
using $u=x^2$, so $\mathrm{d}u = 2x\,\mathrm{d}x$. Then the definite integral is

$$\int_0^{\sqrt{\pi}} x\sin(x^2)\,\mathrm{d}x = \left.\tfrac{-1}{2}\cos(x^2)\right|_{x=0}^{x=\sqrt{\pi}} = \tfrac{-1}{2}(-1)-\tfrac{-1}{2}(1) = \boxed{1}.$$

Alternatively, when $x=0, u=0$ and when $x=\sqrt{\pi}, u=\pi$, so this is

$$\int_0^{\pi}\tfrac12\sin(u)\,\mathrm{d}u = \left.\tfrac{-1}{2}\cos(u)\right|_{u=0}^{u=\pi} = \tfrac{-1}{2}(-1)-\tfrac{-1}{2}(1) = \boxed{1}.$$

(e) $\displaystyle\int \frac{5}{4x+9}\,\mathrm{d}x = \boxed{\frac{5}{4}\ln(4x+9)+C}$

(f) $\displaystyle\int \frac{5x}{4x^2+9}\,\mathrm{d}x = \boxed{\frac{5}{8}\ln(4x^2+9)+C}$

☆ (g) $\displaystyle\int \frac{5}{4x^2+9}\,\mathrm{d}x = \boxed{\frac{5}{6}\arctan(\tfrac{2}{3}x)+C}$

(h) $\displaystyle\int \frac{\sin(\ln(x))}{x}\,\mathrm{d}x = \int \sin(u)\,\mathrm{d}u = -\cos(u)+C = \boxed{-\cos(\ln(x))+C}$

using $u=\ln(x)$, so $\mathrm{d}u = \frac{1}{x}\,\mathrm{d}x$.

☆ (i) $\displaystyle\int_0^9 \sqrt{4-\sqrt{x}}\,\mathrm{d}x$

$u = 4-\sqrt{x}$ gives $\mathrm{d}u = \frac{-1}{2\sqrt{x}}\,\mathrm{d}x$. There is no "$\frac{-1}{2\sqrt{x}}$" in the original integral, but $\mathrm{d}x = -2\sqrt{x}\,\mathrm{d}u$ and $\sqrt{x} = 4-u$, so $\mathrm{d}x = -2(4-u)\,\mathrm{d}u = (2u-8)\,\mathrm{d}u$. Thus

$$\int \sqrt{4-\sqrt{x}}\,\mathrm{d}x = \int \sqrt{u}\,(2u-8)\mathrm{d}u = \int (2u^{3/2}-8u^{1/2})\,\mathrm{d}u$$
$$= \tfrac{4}{5}u^{5/2} - \tfrac{16}{3}u^{3/2} + C.$$

The definite integral is

$$\left(\tfrac{4}{5}(4-x^{1/2})^{5/2} - \tfrac{16}{3}(4-x^{1/2})^{3/2}\right)\Big|_{x=0}^{x=9} = \left(\frac{-68}{15}\right) - \left(\frac{-125}{15}\right) = \boxed{\frac{188}{15}}$$

or, with $4-\sqrt{0}=4$ and $4-\sqrt{9}=1$,

$$\left(\tfrac{4}{5}u^{5/2} - \tfrac{16}{3}u^{3/2}\right)\Big|_{u=4}^{u-1} = \left(\frac{-68}{15}\right) - \left(\frac{-125}{15}\right) = \boxed{\frac{188}{15}}$$

(j) $\displaystyle\int x^3\cos(2x^4)\,\mathrm{d}x = \int \tfrac{1}{8}\cos(u)\,\mathrm{d}u = \tfrac{1}{8}\sin(u)+C = \boxed{\tfrac{1}{8}\sin(2x^4)+C}$

using $u = 2x^4$, so $\mathrm{d}u = 8x^3\,\mathrm{d}x$.

(k) $\displaystyle\int e^{t^5}t^4\,\mathrm{d}t = \int e^u\tfrac{1}{5}\,\mathrm{d}u = \tfrac{1}{5}e^u + C = \boxed{\tfrac{1}{5}e^{t^5}+C}$

using $u = t^5$, so $\mathrm{d}u = 5t^4\,\mathrm{d}t$.

(l) $\displaystyle\int \frac{(\ln(x))^2}{5x}\,\mathrm{d}x = \boxed{\frac{(\ln(x))^3}{15}}$

(m) $\displaystyle\int \frac{1}{x\ln(x)}\,\mathrm{d}x = \int \frac{1}{u}\,\mathrm{d}u = \ln(u)+C = \boxed{\ln(\ln(x))+C}$

using $u=\ln(x)$, so $\mathrm{d}u = \frac{1}{x}\,\mathrm{d}x$.

(n) $\displaystyle\int_0^{\pi/2} \sin(x)\cos(x)\,\mathrm{d}x = \boxed{\frac{1}{2}}$

(o) $\displaystyle\int \sin(1-x)(2-\cos(1-x))^4\,\mathrm{d}x = \boxed{\frac{-1}{5}(2-\cos(1-x))^5+C}$

(p) $\displaystyle\int (1-\tfrac{1}{v})\cos(v-\ln(v))\,\mathrm{d}v = \boxed{\sin(v-\ln(v))+C}$

(q) $\displaystyle\int \frac{t}{\sqrt{1-4t^2}}\,\mathrm{d}t = \boxed{\tfrac{-1}{4}\sqrt{1-4t^2}+C}$

(r) $\displaystyle\int_0^{\pi/3} \big(3\sin(\tfrac{1}{2}x)+5\cos(x)\big)\,\mathrm{d}x = \boxed{6-\frac{\sqrt{3}}{2}}$

(s) $\displaystyle\int \frac{e^{\tan(x)}}{\cos(x)^2}\,\mathrm{d}x = \boxed{e^{\tan(x)}+C}$

(t) $\displaystyle\int_1^5 \frac{x^2+1}{x^3+3x}\,\mathrm{d}x = \tfrac{1}{3}\ln(x^3+3x)\Big|_1^5 = \boxed{\frac{1}{3}\ln(35)}$

143. If $\displaystyle\int_9^{16} f(x)\,\mathrm{d}x = 1$, calculate $\displaystyle\int_3^{10} f(x^2)\,x\,\mathrm{d}x = \boxed{\frac{1}{2}}$

144. If $\displaystyle\int_0^1 f(x)\,\mathrm{d}x = 19$, calculate each of the following or say that there is not enough information to possibly do the calculation.

(a) $\displaystyle\int_0^1 f(x^5)\,5x^4\,\mathrm{d}x = \boxed{19}$ (b) $\displaystyle\int_0^1 f(x^5)\,x^4\,\mathrm{d}x = \boxed{\frac{19}{5}}$

(c) $\displaystyle\int_0^1 f(\tfrac{1}{5}x^5)\,x^4\,\mathrm{d}x$ $\boxed{\text{not enough info}}$

(d) $\displaystyle\int_0^1 \frac{f(\sqrt{x}\,)}{\sqrt{x}}\,\mathrm{d}x = \boxed{38}$

(e) $\displaystyle\int_0^1 f(\sin(x))\cos(x)\,\mathrm{d}x$ $\boxed{\text{not enough info}}$

(f) $\displaystyle\int_0^1 f(\sin(\tfrac{\pi}{2}x))\cos(\tfrac{\pi}{2}x)\,\mathrm{d}x = \boxed{\frac{38}{\pi}}$

145. Fill in the missing parts of the table:

$f =$	$\sin(x)$	$\ln(x)$	x^3	$\boxed{\frac{1}{2}x^2}$	$\boxed{\ln(x)}$	$\boxed{-\cos(x)}$
$\mathrm{d}f =$	$\cos(x)\,\mathrm{d}x$	$\boxed{\frac{1}{x}\,\mathrm{d}x}$	$\boxed{3x^2\,\mathrm{d}x}$	$x\,\mathrm{d}x$	$\frac{\mathrm{d}x}{x}$	$\sin(x)\,\mathrm{d}x$

146. Find the derivative of $2xe^{2x}$.

Integration by parts for indefinite integrals:

$$\int u\,dv = uv - \int v\,\mathrm{d}u.$$

147. Use integration by parts with $u = 4x$ and $dv = e^{2x}\,\mathrm{d}x$ to evaluate $\displaystyle\int 4xe^{2x}\,\mathrm{d}x$.

$\boxed{(2x-1)e^{2x}+C}$

148. Use integration by parts with $u = \ln(x)$ and $dv = 1\,\mathrm{d}x$ to find $\displaystyle\int \ln(x)\,\mathrm{d}x$.

$\mathrm{d}u = \frac{1}{x}\,\mathrm{d}x$, and $v = x$, so

$$\int \ln(x)1\,\mathrm{d}x + \int x\frac{1}{x}\,\mathrm{d}x = x\ln(x)$$
$$\int \ln(x)\,\mathrm{d}x + \int 1\,\mathrm{d}x = x\ln(x)$$
$$\int \ln(x)\,\mathrm{d}x + x + C = x\ln(x)$$
$$\int \ln(x)\,\mathrm{d}x = \boxed{x\ln(x) - x + C}$$

149. Find the following indefinite integrals using integration by parts:

(a) $\int x\sin(x)\,\mathrm{d}x$

With $u = x$ and $\mathrm{d}v = \sin(x)\,\mathrm{d}x$, we have $\mathrm{d}u = \mathrm{d}x$ and $v = -\cos(x)$.

$$uv - \int v\,\mathrm{d}u = x(-\cos(x)) + \int \cos(x)\,\mathrm{d}x = \boxed{\sin(x) - x\cos(x) + C}.$$

(b) $\int x\cos(8x)\,\mathrm{d}x = \boxed{\frac{1}{8}\,x\sin(8x) + \frac{1}{64}\cos(8x) + C}$

(c) $\int \frac{\ln(x)}{x^5}\,\mathrm{d}x = \boxed{-\frac{\ln(x)}{4x^4} - \frac{1}{16x^4} + C}$

(d) $\int x^2\cos(4x)\,\mathrm{d}x$

With $u = x^2$ and $\mathrm{d}v = \cos(4x)\,\mathrm{d}x$, we have $\mathrm{d}u = 2x\,\mathrm{d}x$ and $v = \frac{1}{4}\sin(4x)$.

$$\begin{aligned}\int x^2\cos(4x)\,\mathrm{d}x = uv - \int v\,\mathrm{d}u &= \tfrac{1}{4}x^2\sin(4x) - \int \tfrac{1}{4}\sin(4x)2x\,\mathrm{d}x \\ &= \tfrac{1}{4}x^2\sin(4x) - \tfrac{1}{2}\int x\sin(4x)\,\mathrm{d}x\end{aligned}$$

This new integral also requires integration by parts (it is extremely similar to parts a and b).

New $u = x$ and $\mathrm{d}v = \sin(4x)\,\mathrm{d}v$ gives $\mathrm{d}u = \mathrm{d}x$ and $v = \frac{-1}{4}\cos(4x)$, so

$$\begin{aligned}\int x\sin(4x)\,\mathrm{d}x &= x(\tfrac{-1}{4}\cos(4x)) - \int \tfrac{-1}{4}\cos(4x)\,\mathrm{d}x \\ &= \tfrac{-1}{4}x\cos(4x) - \tfrac{1}{16}\sin(4x) + C\end{aligned}$$

and then

$$\begin{aligned}\int x^2\cos(4x)\,\mathrm{d}x &= \tfrac{1}{4}x^2\sin(4x) - \tfrac{1}{2}\left(\tfrac{-1}{4}x\cos(4x) - \tfrac{1}{16}\sin(4x) + C\right) \\ &= \tfrac{1}{4}x^2\sin(4x) + \tfrac{1}{8}x\cos(4x) - \tfrac{1}{32}\sin(4x) + C \\ &= \boxed{\tfrac{1}{8}\,x\cos(4x) + (\tfrac{1}{4}x^2 - \tfrac{1}{32})\sin(4x) + C}\end{aligned}$$

(e) $\int (4x+12)e^{x/3}\,\mathrm{d}x$

$u = 4x+12$ and $\mathrm{d}v = e^{x/3}\,\mathrm{d}x$ gives $\mathrm{d}u = 4\,\mathrm{d}x$ and $v = 3e^{x/3}$.

$(4x+12)(3e^{x/3}) - \int 12e^{x/3}\,\mathrm{d}x = (12x+36)e^{x/3} - 36e^{x/3} + C = \boxed{12x\,e^{x/3} + C}$.

(f) $\int \cos(x)e^{2x}\,\mathrm{d}x$

$u = \cos(x)$ and $\mathrm{d}v = e^{2x}\,\mathrm{d}x$ gives $\mathrm{d}u = -\sin(x)\,\mathrm{d}x$ and $v = \frac{1}{2}e^{2x}$.

$$\begin{aligned}\int e^{2x}\cos(x)\,\mathrm{d}x &= \tfrac{1}{2}\cos(x)e^{2x} - \int \tfrac{-1}{2}e^{2x}\sin(x)\,\mathrm{d}x\\ &= \tfrac{1}{2}\cos(x)e^{2x} + \tfrac{1}{2}\int e^{2x}\sin(x)\,\mathrm{d}x.\end{aligned}$$

We need parts again to do $\int e^{2x}\sin(x)\,\mathrm{d}x$.

New $u = \sin(x)$ and $\mathrm{d}v = e^{2x}$ (again) gives $\mathrm{d}u = \cos(x)\,\mathrm{d}x$ and $v = \frac{1}{2}e^{2x}$.

$$\int e^{2x}\sin(x)\,\mathrm{d}x = \tfrac{1}{2}\sin(x)e^{2x} - \int \tfrac{1}{2}e^{2x}\cos(x)\,\mathrm{d}x.$$

Therefore

$$\begin{aligned}\int e^{2x}\cos(x)\,\mathrm{d}x &= \tfrac{1}{2}\cos(x)e^{2x} + \tfrac{1}{2}\left(\tfrac{1}{2}\sin(x)e^{2x} - \int \tfrac{1}{2}e^{2x}\cos(x)\,\mathrm{d}x\right)\\ \int \boldsymbol{e^{2x}\cos(x)\,\mathrm{d}x} &= \tfrac{1}{2}\cos(x)e^{2x} + \tfrac{1}{4}\sin(x)e^{2x} - \tfrac{1}{4}\int \boldsymbol{e^{2x}\cos(x)\,\mathrm{d}x}\\ \tfrac{5}{4}\int \boldsymbol{e^{2x}\cos(x)\,\mathrm{d}x} &= \tfrac{1}{2}\cos(x)e^{2x} + \tfrac{1}{4}\sin(x)e^{2x} + C\\ \int \boldsymbol{e^{2x}\cos(x)\,\mathrm{d}x} &= \boxed{\tfrac{2}{5}\cos(x)e^{2x} + \tfrac{1}{5}\sin(x)e^{2x} + C}\end{aligned}$$

150. Calculate the following definite integrals using integration by parts:

(a) $\int_0^6 (4x+12)e^{x/3}\,\mathrm{d}x$ We could use **Task 149(e)** and then

$$12x\,e^{x/3}\Big|_0^6 = 72\,e^2 - 0 = \boxed{72\,e^2}.$$

Alternatively, we can use the formula from the box, with $u = 4x+12$ and $v = 3e^{x/3}$:

$$\begin{aligned}\int_0^6 (4x+12)3e^{x/3}\,\mathrm{d}x + \int_0^6 3e^{x/3}4\,\mathrm{d}x &= (4x+12)3e^{x/3}\Big|_{x=0}^{x=6}\\ \int_0^6 (4x+12)3e^{x/3}\,\mathrm{d}x + 36e^{x/3}\Big|_{x=0}^{x=6} &= (4x+12)3e^{x/3}\Big|_{x=0}^{x=6}\\ \int_0^6 (4x+12)3e^{x/3}\,\mathrm{d}x + 36e^2 - 36 &= 108e^2 - 36\\ \int_0^6 (4x+12)3e^{x/3}\,\mathrm{d}x &= 108e^2 - 36e^2 = \boxed{72e^2}\end{aligned}$$

(b) $\displaystyle\int_1^2 x\ln(x)\,\mathrm{d}x = \boxed{\ln(4) - \frac{3}{4}}$

(c) $\displaystyle\int_0^1 t\sin(\pi t)\,\mathrm{d}t = \boxed{\frac{1}{\pi}}$

(d) $\displaystyle\int_0^\pi x^4\cos(4x)\,\mathrm{d}x = \boxed{\frac{\pi}{8}}$

☆151. Prove that $\displaystyle\int_1^\pi \ln(x)\cos(x)\,\mathrm{d}x = \int_1^\pi \frac{-\sin(x)}{x}\,\mathrm{d}x.$

$$\int_1^\pi \ln(x)\cos(x)\,\mathrm{d}x + \int_1^\pi \frac{\sin(x)}{x}\,\mathrm{d}x = \ln(x)\sin(x)\Big|_{x=0}^{x=\pi} = 0$$

☆152. If $g(0) = 8$, $g(1) = 5$, and $\displaystyle\int_0^1 g(x)\,\mathrm{d}x = 2$, find the value of $\displaystyle\int_0^1 xg'(x)\,\mathrm{d}x$.

Parts with $u = x$ and $\mathrm{d}v = g'(x)\,\mathrm{d}x$ gives $\mathrm{d}u = \mathrm{d}x$ and $v = g(x)$, so as indefinite integrals,

$$\int xg'(x)\,\mathrm{d}x = \int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u = xg(x) - \int g(x)\,\mathrm{d}x.$$

As definite integrals,

$$\int_0^1 xg'(x)\,\mathrm{d}x = xg(x)\Big|_0^1 - \int_0^1 g(x)\,\mathrm{d}x = 1\cdot g(1) - 0\cdot g(0) - 2 = 5 - 2 = \boxed{3}.$$

153. Try each of the following methods to find $\displaystyle\int \sin(x)\cos(x)\,\mathrm{d}x$. (They are all possible.)

(a) Substitue $u = \sin(x)$, so $\mathrm{d}u = \cos(x)\,\mathrm{d}x$ and the integral is $\displaystyle\int u\,\mathrm{d}u$.

(b) Substitue $u = -\cos(x)$, so $\mathrm{d}u = \sin(x)\,\mathrm{d}x$, and the integral is $\displaystyle\int -u\,\mathrm{d}u$.

(c) Substitute $\sin(x)\cos(x) = \frac{1}{2}\sin(2x)$, so the integral is $\frac{1}{2}\displaystyle\int \sin(2x)\,\mathrm{d}x$.

(d) Do integration by parts with $u = \sin(x)$ and $\mathrm{d}v = \cos(x)\,\mathrm{d}x$.

(e) Do integration by parts with $u = \cos(x)$ and $\mathrm{d}v = \sin(x)\,\mathrm{d}x$.

☆(f) Compare your answers to parts (a) - (e).
See `https://youtu.be/-JR9-dgU7tU?t=520`

154. Find $\displaystyle\int 4x\cos(2-3x)\,\mathrm{d}x$ and $\displaystyle\int (2-3x)\cos(4x)\,\mathrm{d}x$.

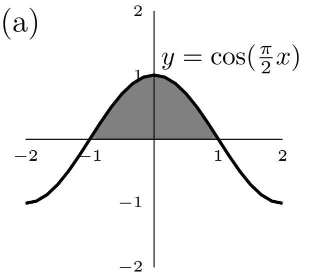
(a) With $u = 4x$ and $\mathrm{d}v = \cos(2-3x)\,\mathrm{d}x$, we have $\mathrm{d}u = 4\,\mathrm{d}x$ and $v = \frac{-1}{3}\sin(2-3x)$.

$$uv - \int v\,\mathrm{d}u = 4x(\tfrac{-1}{3}\sin(2-3x)) - \int 4(\tfrac{-1}{3}\sin(2-3x))\,\mathrm{d}x$$
$$= \boxed{\tfrac{-4}{3}\sin(2-3x) + \tfrac{4}{9}\cos(2-3x) + C}.$$

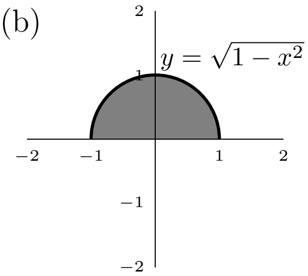
(b) $\boxed{\dfrac{1}{2}\sin(4x) - \dfrac{3}{4}x\sin(4x) - \dfrac{3}{16}\cos(4x) + C}$

155. Give the area of each of the following shapes:

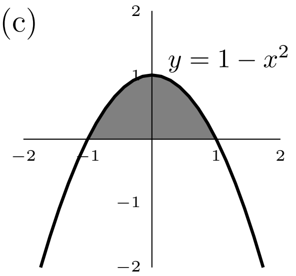
(a)



(b)



(c)



(a) $\displaystyle\int_{-1}^{1} \cos(\tfrac{\pi}{2}x)\,\mathrm{d}x = \tfrac{2}{\pi}\sin(\tfrac{\pi}{2}x)\Big|_{x=-1}^{x=1} = \boxed{\frac{4}{\pi}} \approx 1.2732.$

(b) The area of a semicircle is $\tfrac{1}{2}\pi r^2 = \boxed{\frac{\pi}{2}} \approx 1.5708.$

(c) $\displaystyle\int_{-1}^{1} (1-x^2)\,\mathrm{d}x = (x - \tfrac{1}{3}x^3)\Big|_{x=-1}^{x=1} = \boxed{\frac{4}{3}} \approx 1.3333.$

156. Calculate each of the following integrals.
Some* require substitution, some** require parts, and some do not need either.

(a) $\displaystyle\int (x^4 + x^{1/2} + 4 + x^{-1})\,\mathrm{d}x = \boxed{\tfrac{1}{5}x^5 + \tfrac{1}{3}x^{3/2} + \ln|x| + C}$

(b) $\displaystyle\int \left(x^2 + \sqrt{x} + \frac{\ln(81)}{\ln(3)} + \frac{1}{x}\right)\mathrm{d}x = \boxed{\tfrac{1}{5}x^5 + \tfrac{1}{3}x^{3/2} + \ln|x| + C}$

(c) $\displaystyle\int (t + e^t)\,\mathrm{d}t = \boxed{\frac{t^2}{2} + e^t + C}$

(d) $\displaystyle\int (t \cdot e^t)\,\mathrm{d}t = \boxed{(t-1)e^t + C}$

(e) $\displaystyle\int (t^3 + e^{3t})\,\mathrm{d}t = \boxed{\frac{t^4}{4} + \frac{e^{3t}}{3} + C}$

☆ (f) $\displaystyle\int (t^3 \cdot e^{3t})\,\mathrm{d}t = \boxed{\tfrac{1}{27}e^{3t}(9t^3 - 9t^2 + 6t - 2) + C}$

(g) $\displaystyle\int \frac{x}{x^2+1}\,\mathrm{d}x = \boxed{\tfrac{1}{2}\ln(x^2+1) + C}$

(h) $\displaystyle\int \frac{x}{x^2-1}\,\mathrm{d}x = \boxed{\tfrac{1}{2}\ln|x^2-1| + C}$

(i) $\displaystyle\int \frac{x^2-1}{x}\,\mathrm{d}x = \boxed{\tfrac{1}{2}x^2 - \ln|x| + C}$

(j) $\displaystyle\int \frac{1}{x^2-1}\,\mathrm{d}x = \boxed{\tfrac{1}{2}\ln|1-x| - \tfrac{1}{2}\ln|x+1| + C}$

☆ (k) $\displaystyle\int \frac{1}{x^2+1}\,\mathrm{d}x = \boxed{\arctan(x) + C}$

(ℓ) $\displaystyle\int \frac{y}{\sqrt{y^2+1}}\,\mathrm{d}y = \boxed{\sqrt{y^2+1} + C}$

☆ (m) $\displaystyle\int \frac{1}{\sqrt{y^2+1}}\,\mathrm{d}y = \boxed{\ln(y+\sqrt{y^2+1})+C}$

(n) $\displaystyle\int t\ln(t)\,\mathrm{d}t = \boxed{\tfrac{1}{2}t^2\ln(t) - \tfrac{1}{4}t^4 + C}$

(o) $\displaystyle\int \frac{3t-12}{\sqrt{t^2-8t+6}}\,\mathrm{d}t = \boxed{3\sqrt{t^2-8t+6}+C}$

(p) $\displaystyle\int \frac{1}{\sqrt{x-1}}\,\mathrm{d}x = \boxed{2\sqrt{x-1}+C}$

(q) $\displaystyle\int \frac{x}{\sqrt{x-1}}\,\mathrm{d}x = \boxed{\tfrac{2}{3}(x+2)\sqrt{x-1}+C}$

(r) $\displaystyle\int y^3\,\mathrm{d}y = \boxed{\tfrac{1}{4}y^4+C}$

(s) $\displaystyle\int y(y+1)(y-1)\,\mathrm{d}y = \boxed{\tfrac{1}{4}y^4 - \tfrac{1}{2}y^2 + C}$

(t) $\displaystyle\int x\sin(2x)\,\mathrm{d}x = \boxed{\frac{1}{4}\sin(2x) - \frac{1}{2}x\cos(2x) + C}$

(u) $\displaystyle\int x^3\sin(2x^4)\,\mathrm{d}x = \boxed{\tfrac{-1}{8}\cos(2x^4)+C}$

☆ (v) $\displaystyle\int x^7\sin(2x^4)\,\mathrm{d}x$

This can be done using parts and substitution together!

If we substitute $w = 2x^4$ then $\mathrm{d}w = 8x^3\,\mathrm{d}x$ and, using $x^7 = x^4\cdot x^3$, we have

$$\begin{aligned}\int x^7\sin(2x^4)\,\mathrm{d}x &= \int x^4\cdot\sin(2x^4)\cdot x^3\,\mathrm{d}x\\ &= \int(\tfrac{1}{2}w)\cdot\sin(w)\cdot\tfrac{1}{8}\,\mathrm{d}w = \frac{1}{16}\int w\sin(w)\,\mathrm{d}w.\end{aligned}$$

Using integration by parts on $\int w\sin(w)\,\mathrm{d}w$ is exactly like **Task ??(a)**, so we get $\int w\sin(w)\,\mathrm{d}w = \sin(w) - w\cos(w) + C$ and

$$\tfrac{1}{16}\sin(w) - \tfrac{1}{16}w\cos(w) + C = \boxed{\tfrac{1}{16}\sin(2x^4) - \tfrac{1}{8}x^4\cos(2x^4) + C}.$$

☆ (w) $\displaystyle\int \sin(2x^4)\,\mathrm{d}x$

This integral is literally impossible to write nicely. There cannot exist any formula for $F(x)$ using only $+-\times\div$ and compositions of polynomials, trig, exponentials, and logs such that $F'(x) = \sin(2x^4)$.

(x) $\displaystyle\int e^{5x}\cos(e^{5x})\,\mathrm{d}x = \boxed{\tfrac{1}{5}\sin(e^{5x})+C}$

☆ (y) $\displaystyle\int x^5\cos(x)\,\mathrm{d}x = \boxed{(x^5-20x^3+120x)\sin(x) + (5x^4-60x^2+120)\cos(x) + C}$

(z) $\displaystyle\int e^{8\ln(t)}\,\mathrm{d}t = \int t^8\,\mathrm{d}t = \boxed{\tfrac{1}{9}t^9+C}$

* g, h, m, o, p, q, u, x. ** d, f, ℓ, n, t, v, y.

The area between two curves of the form $y = f(x)$ is $\displaystyle\int_{\text{left}}^{\text{right}} \big(\text{top}(x) - \text{bottom}(x)\big)\,\mathrm{d}x$.
The area between two curves of the form $x = g(y)$ is $\displaystyle\int_{\text{bottom}}^{\text{top}} \big(\text{right}(y) - \text{left}(y)\big)\,\mathrm{d}y$.

157. Find the area of the region bounded by $y = e^x, y = x + 5, x = -4$, and $x = 0$ (that is, the area between $y = e^x$ and $y = x + 5$ with $-4 \le x \le 0$).

$$\int_{-4}^{0} \big((x+5) - e^x\big)\,\mathrm{d}x = \boxed{11 + e^{-4}}$$

158. What is the area of the region bounded by the curves $y = 20 - x^4$ and $y = 4$?

$$\int_{-2}^{2} \big((20 - x^4) - 4\big)\,\mathrm{d}x = \boxed{\frac{256}{5}}$$

159. Find the area of the region bounded by the curves $x = y^2$ and $x = 1 + y - y^2$.

$$\int_{-1/2}^{1} \big((1 + y - y^2) - (y^2)\big)\,\mathrm{d}y = \boxed{\frac{9}{8}}$$

160. Calculate the area of...

(a) the region bounded by the curves $y = x^2, y = 4x, x = 2, x = 3$.

$$\int_{2}^{3} \big(4x - x^2\big)\,\mathrm{d}x = \boxed{\frac{11}{3}}$$

(b) the region bounded by the curves $y = x^2, y = 4x, y = 1, y = 4$.

$$\int_{1}^{4} \big(\sqrt{y} - \tfrac{1}{4}y\big)\,\mathrm{d}y = \boxed{\frac{67}{24}}$$

(c) the region bounded by the curves $y = x^2$ and $y = 4x$.

$$\int_{0}^{4} \big(4x - x^2\big)\,\mathrm{d}x = \int_{0}^{16} \big(\sqrt{y} - \tfrac{1}{4}y\big)\,\mathrm{d}y = \boxed{\frac{32}{3}}$$