Analysis 1, Summer 2024

List 5

Definite and indefinite integrals

An **anti-derivative** of $f(x)$ is a function whose derivative is $f(x)$.
In symbols, $F(x)$ is an anti-derivative of $f(x)$ if $F'(x) = f(x)$.

113. (a) Give an anti-derivative of $10x^9$.
That is, give a function $F(x)$ for which $F'(x) = 10x^9$.
(b) Give another anti-derivative of $10x^9$.
(c) Give another anti-derivative of $10x^9$.
(d) Give another anti-derivative of $10x^9$.

114. Give an anti-derivative of $\sin(x)$.

115. Give an anti-derivative for each of the following functions:

(a) x^3
(b) $12x^5$
(c) $12x^4$
(d) x^{15}
(e) $-3x^{15}$
(f) $\frac{1}{2}x^2$
(g) x^{5000}
(h) $\frac{3}{5}x^{12}$
(i) $\frac{-4}{3}x^7$
(j) $5\sin(x)$
(k) $2\cos(x)$
(ℓ) e^x

An **indefinite integral** describes all the anti-derivatives of a function. We write

$$\int f(x)\,dx = F(x) + C,$$

where $F(x)$ is any function for which $F'(x) = f(x)$.

116. Find $\int (2x^5 + 3x - 9)\,\mathrm{d}x$.

117. Find $\int (2u^5 + 3u - 9)\,\mathrm{d}u$.

118. Give each of the following indefinite integrals using basic derivative knowledge:

(a) $\int x^{372.5}\,\mathrm{d}x$
(b) $\int \frac{1}{x}\,\mathrm{d}x$
(c) $\int e^x\,\mathrm{d}x$
(d) $\int 97^x\,\mathrm{d}x$
(e) $\int -\sin(x)\,\mathrm{d}x$
(f) $\int \sin(x)\,\mathrm{d}x$
(g) $\int \cos(x)\,\mathrm{d}x$
(h) $\int 5t^9\,\mathrm{d}t$

119. If $u = 6x^2 - 5$, give a formula for $\mathrm{d}u$ (this formula will have x and $\mathrm{d}x$ in it) and a formula for $\mathrm{d}x$ (this formula will have x and $\mathrm{d}u$ in it).

The notation $\boldsymbol{g(x)}\Big|_{\boldsymbol{x=a}}^{\boldsymbol{x=b}}$ or $g(x)\Big|_a^b$ means to do the subtraction $g(b) - g(a)$.

120. Calculate $\frac{1}{3}x^3\Big|_{x=1}^{x=3}$.

121. Calculate $(x^3 + \frac{1}{2}x)\Big|_{x=1}^{x=5}$.

122. Calculate $\frac{1-x}{e^x}\Big|_{x=0}^{x=1}$.

The **definite integral** $\int_a^b f(x)\,dx$, spoken as "the integral from a to b of $f(x)$ with respect to x", is the (signed) area of the region with $x = a$ on the left, $x = b$ on the right, $y = 0$ at the bottom, and $y = f(x)$ at the top (but if $f(x) < 0$ for some x or if $b < a$ then it's possible for the area to be negative).
The **Fundamental Theorem of Calculus** says that

$$\int_a^b f(x)\,\mathrm{d}x \;=\; F(x)\Big|_{x=a}^{x=b} \;=\; F(b) - F(a),$$

where $F(x)$ is any function for which $F'(x) = f(x)$.

123. Calculate $\int_1^3 x^2\,\mathrm{d}x$ using the FTC.

124. Write, in symbols, the integral from zero to six of x^2 with respect to x, then find the value of that definite integral.

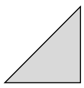
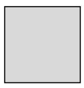
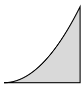
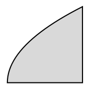
125. Evaluate (meaning find of the value of) the following definite integrals using common area formulas.

(a) $\int_3^9 2\,\mathrm{d}x$ (d) $\int_{-2}^4 |x|\,\mathrm{d}x$ (g) $\int_1^5 3x\,\mathrm{d}x$

(b) $\int_3^9 -2\,\mathrm{d}x$ (e) $\int_{-2}^4 x\,\mathrm{d}x$ (h) $\int_{-4}^4 \sqrt{16 - x^2}\,\mathrm{d}x$

(c) $\int_0^5 x\,\mathrm{d}x$ (f) $\int_0^5 3x\,\mathrm{d}x$ (i) $\int_0^7 \sqrt{49 - x^2}\,\mathrm{d}x$

126. Match the shapes (a)-(d) with the integral (I)-(IV) that is most likely to calculate its area.

(a)  (b)  (c)  (d) 

(I) $\int_0^1 \sqrt{x}\,\mathrm{d}x$ (II) $\int_0^1 x\,\mathrm{d}x$ (III) $\int_0^1 x^2\,\mathrm{d}x$ (IV) $\int_0^1 1\,\mathrm{d}x$

127. Find $\int_0^1 \sqrt{x}\,\mathrm{d}x$.

128. Evaluate the following definite integrals using the FTC. Your answer for each should be a number.

(a) $\int_{-3}^9 2\,\mathrm{d}x$ (d) $\int_0^9 (x^3 - 9x)\,\mathrm{d}x$ (g) $\int_0^1 (e^x + x^e)\,\mathrm{d}x$ (j) $\int_9^9 \sin(x^2)\,\mathrm{d}x$

(b) $\int_1^5 3x\,\mathrm{d}x$ (e) $\int_0^\pi \sin(t)\,\mathrm{d}t$ (h) $\int_{-1}^1 x^2\,\mathrm{d}x$ (k) $\int_0^5 \cos(x)\,\mathrm{d}x$

(c) $\int_1^{12} \frac{1}{x}\,\mathrm{d}x$ (f) $\int_2^8 3 \cdot \sqrt{u}\,\mathrm{d}u$ (i) $\int_1^3 t\,\mathrm{d}t$

129. Evaluate the following definite integrals using the FTC. Your answers will be formulas.

(a) $\displaystyle\int_a^9 2\,\mathrm{d}x$ (b) $\displaystyle\int_1^5 kx\,\mathrm{d}x$

(c) $\displaystyle\int_1^t \frac{1}{x}\,\mathrm{d}x$ assuming $t > 1$

(d) $\displaystyle\int_0^9 (x^p - qx)\,\mathrm{d}x$ assuming $p > -1$

130. If $\displaystyle\int_1^4 f(x)\,\mathrm{d}x = 12$ and $\displaystyle\int_1^6 f(x)\,\mathrm{d}x = 15$, what is the value of $I = \displaystyle\int_4^6 f(x)\,\mathrm{d}x$?

131. If $\displaystyle\int_0^1 f(x)\,\mathrm{d}x = 7$ and $\displaystyle\int_0^1 g(x)\,\mathrm{d}x = 3$, calculate each of the following or say that there is not enough information to possibly do the calculation.

(a) $\displaystyle\int_0^1 \big(f(x) + g(x)\big)\,\mathrm{d}x$ (c) $\displaystyle\int_0^1 \big(f(x)\cdot g(x)\big)\,\mathrm{d}x$ (e) $\displaystyle\int_0^1 \big(f(x)^5\big)\,\mathrm{d}x$

(b) $\displaystyle\int_0^1 \big(f(x) - g(x)\big)\,\mathrm{d}x$ (d) $\displaystyle\int_0^1 \big(5f(x)\big)\,\mathrm{d}x$

132. Simplify $\displaystyle\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_1^t \frac{1}{x}\,\mathrm{d}x\right)$ to a formula that does not include x (assume $t > 1$).

133. Simplify $\displaystyle\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_3^t \frac{\sin(x)}{x}\,\mathrm{d}x\right)$ to a formula that does not include x (assume $t > 3$).

134. Simplify $\displaystyle\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^{t^2} \sin(x)\,\mathrm{d}x\right)$ and $\displaystyle\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^{t^2} \sin(x^2)\,\mathrm{d}x\right)$ to formulas that do not include x.

135. Given that $\displaystyle\int \ln(x)\,\mathrm{d}x = x\ln(x) - x + C$, evaluate $\displaystyle\int_1^{e^5} \ln(x)\,\mathrm{d}x$.

Substitution: $\displaystyle\int f(u(x))\cdot u'(x)\,\mathrm{d}x = \int f(u)\,\mathrm{d}u$

136. (a) Re-write $\displaystyle\int \frac{x}{(6x^2-5)^3}\,\mathrm{d}x$ as $\displaystyle\int \ldots\,\mathrm{d}u$ using the substitution $u = 6x^2 - 5$.

(b) Find $\displaystyle\int \frac{x}{(6x^2-5)^3}\,\mathrm{d}x$. (Your final answer should not have u at all.)

137. (a) Re-write $\displaystyle\int x^3\sin(x^4)\,\mathrm{d}x$ as $\displaystyle\int \ldots\,\mathrm{d}u$ using the substitution $u = x^4$.

(b) Find $\displaystyle\int x^3\sin(x^4)\,\mathrm{d}x$.

138. (a) Re-write $\displaystyle\int x\sin(x^4)\,\mathrm{d}x$ as $\displaystyle\int \ldots\,\mathrm{d}u$ using the substitution $u = x^2$.

☆(b) Find $\displaystyle\int x\sin(x^4)\,\mathrm{d}x$.

139. Find $\displaystyle\int \frac{x^4 - x^3 - 1}{4x^5 - 5x^4 - 20x + 3}\,\mathrm{d}x$ using substitution.

140. Find $\displaystyle\int \cot(x)\,\mathrm{d}x = \int \frac{\cos(x)}{\sin(x)}\,\mathrm{d}x$ using substitution.

141. Which of the following has the same value as $\displaystyle\int_2^4 \frac{3x^2 - 2}{\ln(x^3 - 2x + 1)}\,\mathrm{d}x$?

(A) $\displaystyle\int_5^{57} \frac{1}{\ln(u)}\,\mathrm{d}u$ (B) $\displaystyle\int_2^{4} \frac{1}{\ln(u)}\,\mathrm{d}u$ (C) $\displaystyle\int_{10}^{46} \frac{1}{\ln(u)}\,\mathrm{d}u$ (D) $\displaystyle\int_1^{2} \frac{1}{\ln(u)}\,\mathrm{d}u$

142. Find the following integrals using substitution:

(a) $\displaystyle\int (5 - x)^{10}\,\mathrm{d}x$

(b) $\displaystyle\int_1^3 \frac{x}{(6x^2 - 5)^3}\,\mathrm{d}x$

(c) $\displaystyle\int \sqrt{4x + 3}\,\mathrm{d}x$

(d) $\displaystyle\int_0^{\sqrt{\pi}} x\sin(x^2)\,\mathrm{d}x$

(e) $\displaystyle\int \frac{5}{4x + 9}\,\mathrm{d}x$

(f) $\displaystyle\int \frac{5x}{4x^2 + 9}\,\mathrm{d}x$

☆ (g) $\displaystyle\int \frac{5}{4x^2 + 9}\,\mathrm{d}x$

(h) $\displaystyle\int \frac{\sin(\ln(x))}{x}\,\mathrm{d}x$

☆ (i) $\displaystyle\int_0^9 \sqrt{4 - \sqrt{x}}\,\mathrm{d}x$

(j) $\displaystyle\int x^3\cos(2x^4)\,\mathrm{d}x$

(k) $\displaystyle\int e^{t^5} t^4\,\mathrm{d}t$

(l) $\displaystyle\int \frac{(\ln(x))^2}{5x}\,\mathrm{d}x$

(m) $\displaystyle\int \frac{1}{x\ln(x)}\,\mathrm{d}x$

(n) $\displaystyle\int_0^{\pi/2} \sin(x)\cos(x)\,\mathrm{d}x$

(o) $\displaystyle\int \sin(1 - x)(2 - \cos(1 - x))^4\,\mathrm{d}x$

(p) $\displaystyle\int (1 - \tfrac{1}{v})\cos(v - \ln(v))\,\mathrm{d}v$

(q) $\displaystyle\int \frac{t}{\sqrt{1 - 4t^2}}\,\mathrm{d}t$

(r) $\displaystyle\int_0^{\pi/3} \left(3\sin(\tfrac{1}{2}x) + 5\cos(x)\right)\mathrm{d}x$

(s) $\displaystyle\int \frac{e^{\tan(x)}}{\cos(x)^2}\,\mathrm{d}x$

(t) $\displaystyle\int_1^5 \frac{x^2 + 1}{x^3 + 3x}\,\mathrm{d}x$

143. If $\displaystyle\int_9^{16} f(x)\,\mathrm{d}x = 1$, calculate $\displaystyle\int_3^{10} f(x^2)\,x\,\mathrm{d}x$.

144. If $\displaystyle\int_0^1 f(x)\,\mathrm{d}x = 19$, calculate each of the following or say that there is not enough information to possibly do the calculation.

(a) $\displaystyle\int_0^1 f(x^5)\,5x^4\,\mathrm{d}x$

(b) $\displaystyle\int_0^1 f(x^5)\,x^4\,\mathrm{d}x$

(c) $\displaystyle\int_0^1 f(\tfrac{1}{5}x^5)\,x^4\,\mathrm{d}x$

(d) $\displaystyle\int_0^1 \frac{f(\sqrt{x})}{\sqrt{x}}\,\mathrm{d}x$

(e) $\displaystyle\int_0^1 f(\sin(x))\cos(x)\,\mathrm{d}x$

(f) $\displaystyle\int_0^1 f(\sin(\tfrac{\pi}{2}x))\cos(\tfrac{\pi}{2}x)\,\mathrm{d}x$

145. Fill in the missing parts of the table:

$f =$	$\sin(x)$	$\ln(x)$	x^3			
$\mathrm{d}f =$	$\cos(x)\,\mathrm{d}x$			$x\,\mathrm{d}x$	$\frac{\mathrm{d}x}{x}$	$\sin(x)\,\mathrm{d}x$

146. Find the derivative of $2xe^{2x}$.

Integration by parts for indefinite integrals:

$$\int u\,dv = uv - \int v\,\mathrm{d}u.$$

147. Use integration by parts with $u = 4x$ and $\mathrm{d}v = e^{2x}\,\mathrm{d}x$ to evaluate $\int 4xe^{2x}\,\mathrm{d}x$.

148. Use integration by parts with $u = \ln(x)$ and $\mathrm{d}v = 1\,\mathrm{d}x$ to find $\int \ln(x)\,\mathrm{d}x$.

149. Find the following indefinite integrals using integration by parts:

(a) $\int x\sin(x)\,\mathrm{d}x$ (c) $\int \frac{\ln(x)}{x^5}\,\mathrm{d}x$ (e) $\int (4x+12)e^{x/3}\,\mathrm{d}x$

(b) $\int x\cos(8x)\,\mathrm{d}x$ (d) $\int x^2\cos(4x)\,\mathrm{d}x$ (f) $\int \cos(x)e^{2x}\,\mathrm{d}x$

150. Calculate the following definite integrals using integration by parts:

(a) $\int_0^6 (4x+12)e^{x/3}\,\mathrm{d}x$ (c) $\int_0^1 t\sin(\pi t)\,\mathrm{d}t$

(b) $\int_1^2 x\ln(x)\,\mathrm{d}x$ (d) $\int_0^\pi x^4\cos(4x)\,\mathrm{d}x$

☆151. Prove that $\int_1^\pi \ln(x)\cos(x)\,\mathrm{d}x = \int_1^\pi \frac{-\sin(x)}{x}\,\mathrm{d}x$.

☆152. If $g(0) = 8$, $g(1) = 5$, and $\int_0^1 g(x)\,\mathrm{d}x = 2$, find the value of $\int_0^1 xg'(x)\,\mathrm{d}x$.

153. Try each of the following methods to find $\int \sin(x)\cos(x)\,\mathrm{d}x$. (They are all possible.)

(a) Substitue $u = \sin(x)$, so $\mathrm{d}u = \cos(x)\,\mathrm{d}x$ and the integral is $\int u\,\mathrm{d}u$.

(b) Substitue $u = -\cos(x)$, so $\mathrm{d}u = \sin(x)\,\mathrm{d}x$, and the integral is $\int -u\,\mathrm{d}u$.

(c) Substitute $\sin(x)\cos(x) = \frac{1}{2}\sin(2x)$, so the integral is $\frac{1}{2}\int \sin(2x)\,\mathrm{d}x$.

(d) Do integration by parts with $u = \sin(x)$ and $\mathrm{d}v = \cos(x)\,\mathrm{d}x$.

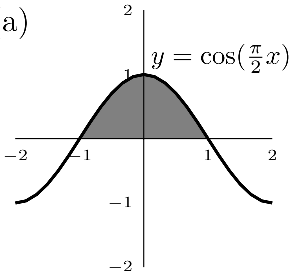
(e) Do integration by parts with $u = \cos(x)$ and $\mathrm{d}v = \sin(x)\,\mathrm{d}x$.

☆(f) Compare your answers to parts (a) - (e).

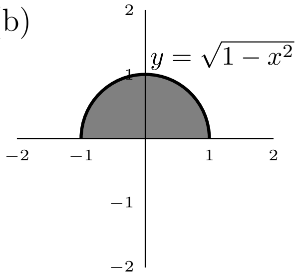
154. Find $\int 4x\cos(2-3x)\,\mathrm{d}x$ and $\int (2-3x)\cos(4x)\,\mathrm{d}x$.

155. Give the area of each of the following shapes:

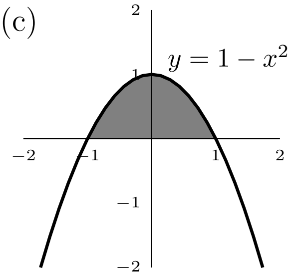
(a)



(b)



(c)



156. Calculate each of the following integrals.
Some* require substitution, some** require parts, and some do not need either.

(a) $\int (x^4+x^{1/2}+4+x^{-1})\,\mathrm{d}x$

(b) $\int \left(x^2+\sqrt{x}+\frac{\ln(81)}{\ln(3)}+\frac{1}{x}\right)\mathrm{d}x$

(c) $\int (t+e^t)\,\mathrm{d}t$

(d) $\int (t\cdot e^t)\,\mathrm{d}t$

(e) $\int (t^3+e^{3t})\,\mathrm{d}t$

☆ (f) $\int (t^3\cdot e^{3t})\,\mathrm{d}t$

(g) $\int \frac{x}{x^2+1}\,\mathrm{d}x$

(h) $\int \frac{x}{x^2-1}\,\mathrm{d}x$

(i) $\int \frac{x^2-1}{x}\,\mathrm{d}x$

(j) $\int \frac{1}{x^2-1}\,\mathrm{d}x$

☆ (k) $\int \frac{1}{x^2+1}\,\mathrm{d}x$

(ℓ) $\int \frac{y}{\sqrt{y^2+1}}\,\mathrm{d}y$

☆ (m) $\int \frac{1}{\sqrt{y^2+1}}\,\mathrm{d}y$

(n) $\int t\ln(t)\,\mathrm{d}t$

(o) $\int \frac{3t-12}{\sqrt{t^2-8t+6}}\,\mathrm{d}t$

(p) $\int \frac{1}{\sqrt{x-1}}\,\mathrm{d}x$

(q) $\int \frac{x}{\sqrt{x-1}}\,\mathrm{d}x$

(r) $\int y^3\,\mathrm{d}y$

(s) $\int y(y+1)(y-1)\,\mathrm{d}y$

(t) $\int x\sin(2x)\,\mathrm{d}x$

(u) $\int x^3\sin(2x^4)\,\mathrm{d}x$

☆ (v) $\int x^7\sin(2x^4)\,\mathrm{d}x$

☆ (w) $\int \sin(2x^4)\,\mathrm{d}x$

(x) $\int e^{5x}\cos(e^{5x})\,\mathrm{d}x$

☆ (y) $\int x^5\cos(x)\,\mathrm{d}x$

(z) $\int e^{8\ln(t)}\,\mathrm{d}t$

* g, h, m, o, p, q, u, x. ** d, f, ℓ, n, t, v, y.

The area between two curves of the form $y = f(x)$ is $\displaystyle\int_{\text{left}}^{\text{right}} \big(\text{top}(x) - \text{bottom}(x)\big)\,\mathrm{d}x$.

The area between two curves of the form $x = g(y)$ is $\displaystyle\int_{\text{bottom}}^{\text{top}} \big(\text{right}(y) - \text{left}(y)\big)\,\mathrm{d}y$.

157. Find the area of the region bounded by $y = e^x, y = x + 5, x = -4$, and $x = 0$ (that is, the area between $y = e^x$ and $y = x + 5$ with $-4 \leq x \leq 0$).

158. What is the area of the region bounded by the curves $y = 20 - x^4$ and $y = 4$?

159. Find the area of the region bounded by the curves $x = y^2$ and $x = 1 + y - y^2$.

160. Calculate the area of...
    (a) the region bounded by the curves $y = x^2, y = 4x, x = 2, x = 3$.
    (b) the region bounded by the curves $y = x^2, y = 4x, y = 1, y = 4$.
    (c) the region bounded by the curves $y = x^2$ and $y = 4x$.