Analysis 1, Summer 2024

List 6

Review for Exam 2

161. Give an equation for the line through the point $(5, 17)$ with slope $\frac{1}{3}$. One possible answer is $\boxed{y - 17 = \frac{1}{3}(x - 5)}$. Other correct answers include $y = \frac{x}{3} + \frac{46}{3}$.

162. Give an equation for the tangent line to $y = \sqrt{x} + x^3$ at $x = 1$.
$\boxed{y = 2 + \frac{7}{2}(x - 1)}$, or $y = \frac{7}{2}x - \frac{3}{2}$.

163. Give an equation for the tangent line to $y = e^{4x\cos x}$ at $x = 0$. $\boxed{y = 4x + 1}$

164. Use the Quotient Rule and the Product Rule to compute $\frac{dy}{dx}$ for $y = \frac{\ln(x)e^x}{x^2}$.
$\frac{x^2(\ln(x)e^x + \frac{1}{x}e^x) - \ln(x)e^x(2x)}{x^4} = \boxed{\frac{e^x}{x^3}\big(1 + (x - 2)\ln(x)\big)}$

165. Calculate the derivative $\ln(5x)$ in two ways:
    (a) Use the rule $\frac{d}{dx}\big[\ln(x)\big] = \frac{1}{x}$ along with the Chain Rule (here $\ln(5x) = f(g(x))$ with $f(x) = \ln(x)$ and $g(x) = 5x$). $\frac{1}{5x} \cdot 5$, which is $\boxed{\frac{1}{x}}$.
    (b) Use algebra to rewrite $\ln(5x) = \ln(x) + \ln(5)$ and then find the derivative of that function. $\frac{1}{x} + 0$, which is $\boxed{\frac{1}{x}}$.

166. On what interval(s) is the function $x^3 - 6x + 11$ increasing? $\boxed{(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)}$ On what interval(s) is it concave up? $\boxed{(0, \infty)}$

167. Find the x-coordinates of all inflection points of $x^4 + 9x^3 - 15x^2 + 17$. $\boxed{x = -5, x = \frac{1}{2}}$

168. Find the x-coordinates of all inflection points of
$$f(x) = \tfrac{3}{10}x^5 - 5x^4 + 32x^3 - 96x^2 + 28.$$
$\boxed{x = 2}$ is the only inflection point. Although $f''(4) = 0$, the sign of f'' *does not change* at $x = 4$ because $f''(x) > 0$ for all x near 4.

169. If f is a smooth function with

x	-2	-1	0	1	2	3	4
f	3	5	-3	7	8	9	12
f'	2	0	-1	-1	1	3	0
f''	0	4	1	-1	$\frac{-8}{3}$	0	1

answer the following:
    (a) Does f have a critical point at $x = 0$? $\boxed{\text{No}}$ $f'(0) \neq 0$
    (b) Does f have a local minimum at $x = -1$? $\boxed{\text{Yes}}$ $f'(-1) = 0$ and $f''(-1) > 0$
    (c) Does f have a local maximum at $x = 4$? $\boxed{\text{No}}$ local min

(d) It it possible that f has an absolute minimum at $x = -1$? $\boxed{\text{No}}$
$f(x) < f(-1)$ for $x = -2$ and $x = 0$

(e) It it possible that f has an absolute maximum at $x = -1$? $\boxed{\text{No}}$
$f(x) > f(-1)$ for $x = 1, 2, \ldots$

(f) It it possible that f has an inflection point at $x = 3$? $\boxed{\text{Yes}}$ $f''(3) = 0$ and $f''(2) < 0 < f''(4)$ (f'' changes sign)

(g) It it possible that f has an inflection point at $x = 4$? $\boxed{\text{No}}$ $f''(4) \neq 0$

170. Find all the critical point(s) of the function

$$f(x) = x^4 - 12x^3 + 30x^2 - 28x$$

and classify each one as a local minimum, local maximum, or neither.

$\boxed{x = -1 \text{ is neither, } x = 7 \text{ is a local min}}$

171. Find all the critical point(s) of the function

$$f(x) = x(6 - x)^{2/3}$$

and classify each one as a local minimum, local maximum, or neither.

After simplifying, $f'(x) = \dfrac{18 - 5x}{3(6 - x)^{1/3}}$, so the critical points are $x = \frac{18}{5}$ (where f' is zero) and $x = 6$ (where f' doesn't exist). The fact that $\boxed{x = \frac{18}{5} \text{ is a local max}}$ can be found from the First or the Second Derivative Test, but the fact that $\boxed{x = 6 \text{ is a local min}}$ requires the First D. Test because $f''(6)$ is not defined.

172. Calculate the following limits, if they exist:

(a) $\displaystyle\lim_{x\to 4} \frac{x^2 - x - 12}{x^2 - 2x - 8} = \boxed{\frac{7}{6}}$ from algebra or L'Hospital

(b) $\displaystyle\lim_{x\to 4} \frac{x^2 + x - 12}{x^2 - 2x - 8}$ $\boxed{\text{doesn't exist}}$

(c) $\displaystyle\lim_{x\to\infty} xe^{-x} = \boxed{0}$ from L'Hospital with $\dfrac{x}{e^x}$

(d) $\displaystyle\lim_{x\to 0^+} x^2 \ln(x) = \boxed{0}$ from L'Hospital with $\dfrac{x^2}{\ln(x)}$

(e) $\displaystyle\lim_{x\to 1} x^2 \ln(x) = \boxed{0}$ just from plugging in $x = 1$

173. Compute $\displaystyle\lim_{x\to 0} (\cos 6x)^{1/x^2}$. Hint: First compute $\displaystyle\lim_{x\to 0} \ln\Big((\cos 6x)^{1/x^2}\Big)$.

$$\ln(a^b) = b\ln(a)$$

$$\ln\Big((\cos 6x)^{1/x^2}\Big) = \frac{1}{x^2}\ln(\cos 6x)$$

$$\begin{aligned}\lim_{x\to 0} \ln\Big((\cos 6x)^{1/x^2}\Big) &= \lim_{x\to 0} \frac{\ln(\cos 6x)}{x^2} \\ &\overset{\text{L'H}}{=} \lim_{x\to 0} \frac{\frac{1}{\cos 6x}\cdot(-6\sin(6x))}{2x} \\ &= \lim_{x\to 0} \frac{-3\sin(6x)}{x\cos(6x)} \\ &\overset{\text{L'H}}{=} \lim_{x\to 0} \frac{-18\cos(6x)}{-6x\sin(x)+\cos(6x)} \\ &= \frac{-18}{0+1} = -18\end{aligned}$$

Since $\ln(\text{Answer}) = -18$, we have Answer $= \boxed{e^{-18}}$.

174. Compute the following indefinite integrals:

(a) $\int 6\,\mathrm{d}x = \boxed{6x + C}$

(b) $\int (2x+6)\,\mathrm{d}x = \boxed{x^2 + 6x + C}$

(c) $\int \frac{8}{x}\,\mathrm{d}x = \boxed{8\ln(x) + C}$

(d) $\int \frac{8}{q}\,\mathrm{d}q = \boxed{8\ln(q) + C}$

(e) $\int x^2\cos(x^3)\,\mathrm{d}x = \boxed{\frac{1}{3}\sin(x^3) + C}$

(f) $\int x^2\cos(x)\,\mathrm{d}x = \boxed{2x\cos x + (x^2-2)\sin x + C}$ using parts twice.

175. Compute the following definite integrals:

(a) $\int_1^5 (2x+6)\,\mathrm{d}x = \boxed{48}$

(b) $\int_0^\pi \frac{1}{3}\sin(u)\,\mathrm{d}u = \boxed{\frac{2}{3}}$

(c) $\int_1^4 (x^3+2x-7)\,\mathrm{d}x = \boxed{\frac{231}{4}}$

(d) $\int_0^\pi 2e^t\sin(5t)\,\mathrm{d}t = \boxed{\frac{5}{13}(e^\pi + 1)}$

176. Find the value of $\int_{-2}^{2} \sqrt{4-x^2}\,\mathrm{d}x$. This is the area of half of a disk with radius $r = 2$, so it is $\frac{1}{2}(\pi r^2) = \frac{1}{2}(4\pi) = \boxed{2\pi}$.

177. Compute the following integrals of rational functions:

(a) $\int \frac{2x+3}{10x^2+30x+40}\,\mathrm{d}x = \boxed{\frac{1}{10}\ln(x^2+3x+4) + C}$

(b) $\displaystyle\int \frac{10x^2+30x+40}{5x}\,\mathrm{d}x = \boxed{x^2+6x+8\ln(x)+C}$

(c) $\displaystyle\int_1^3 \frac{10x^2+30x+40}{5x}\,\mathrm{d}x = \boxed{20+8\ln(3)}$

☆(d) $\displaystyle\int \frac{3}{10x^2+40}\,\mathrm{d}x = \boxed{\tfrac{3}{20}\arctan(\tfrac{x}{2})+C}$ This task is starred.
You will not be asked about inverse tangent on exams in Summer 2024.

178. Find the area of the domain

$$\{(x,y) : 0 \le x \le \pi,\ 0 \le y \le 5\sin(\tfrac{x}{2})\}.$$

$$\int_0^\pi 5\sin(\tfrac{x}{2}) = \Big[-10\cos(\tfrac{x}{2})\Big]_{x=0}^{x=\pi} = \boxed{10}$$

179. Find the area of the domain

$$\{(x,y) : 0 \le x \le \pi,\ 0 \le y \le 2x\sin(3x)+4x\}.$$

$$\int_0^\pi (2x\sin(3x)+4x) = \underbrace{\int_0^\pi 2x\sin(3x)\,\mathrm{d}x}_{\text{parts}} + \underbrace{\int_0^\pi 4x\,\mathrm{d}x}_{\text{basic anti-derivative}} = \boxed{\frac{2}{3}\pi+2\pi^2}$$

180. Find the area of the region bounded by the curves $y = x^2$ and $y = 10 - x^2$.

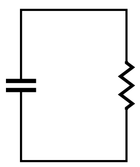
$$\int_{-\sqrt{5}}^{\sqrt{5}} \left((10-x^2)-x^2\right)\mathrm{d}x = \boxed{\frac{40\sqrt{5}}{3}}$$

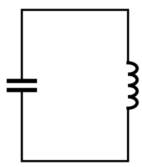
181. Calculate the area of the region bounded by $x = 1$, $y = 1$, and $y = \ln(x)$.

Option 1: $\displaystyle\int_1^e (\text{top} - \text{bottom})\,\mathrm{d}x = \int_1^e (1-\ln(x))\,\mathrm{d}x = \boxed{e-2}$

Option 2: $\displaystyle\int_0^1 (\text{right} - \text{left})\,\mathrm{d}y = \int_0^1 (e^y - 1)\,\mathrm{d}y = \boxed{e-2}$

☆182. Below are two circuits and an equation involving a derivative for each of them. Here R (resistance), C (capacitance), and L (inductance) are all constants, but the voltage $V = V(t)$ is a function of time.

 $V + RC\dfrac{\mathrm{d}V}{\mathrm{d}t} = 0$

 $V + LC\dfrac{\mathrm{d}^2V}{\mathrm{d}t^2} = 0$

(a) Could the first circuit have $V(t) = 5e^{-t/12}$? $\boxed{\text{yes}}$ The second circuit? $\boxed{\text{no}}$

(b) Could the first circuit have $V(t) = 5(1-e^{-t/12})$? $\boxed{\text{no}}$ The second circuit? $\boxed{\text{no}}$

(c) Could the first circuit have $V(t) = 5\sin(2\pi t)$? $\boxed{\text{no}}$ The second circuit? $\boxed{\text{yes}}$

(d) Could the first circuit have $V(t) = 0$? $\boxed{\text{yes}}$ The second circuit? $\boxed{\text{yes}}$

☆183. The "information entropy" h of a weighted coin where Heads (Orzel) has probability p_1 and Tails (Reszka) has probability p_2 is

$$-p_1 \ln(p_1) - p_2 \ln(p_2).$$

Labeling $x = p_1$, we have $p_2 = 1 - x$ (because probabilities must add to 1), so the entropy of a coin with $P(\text{Heads}) = x$ is

$$h(x) \quad = \quad -x \ln(x) - (1-x)\ln(1-x).$$

Determine the x-value that gives the maximum of $h(x)$.
(This will be the *most random* kind of coin.)

For $f(x) = -x\ln(x) - (1-x)\ln(1-x)$, we get $f'(x) = \ln(1-x) - \ln(x)$ after simplification (this requires the chain rule for $\frac{\mathrm{d}}{\mathrm{d}x}\ln(1-x) = \frac{-1}{1-x}$). The maximum of this function occurs when

$$\begin{aligned} f'(x) &= 0 \\ \ln(1-x) - \ln(x) &= 0 \\ \ln(1-x) &= \ln(x) \\ 1 - x &= x \\ 1 &= 2x \\ \boxed{x = \tfrac{1}{2}} \end{aligned}$$

184. Give the one hundredth derivative of $x\,e^x$, that is, $\dfrac{\mathrm{d}^{100}}{\mathrm{d}x^{100}}\left[x\,e^x\right]$. $\boxed{xe^x + 100e^x}$

185. For the function

$$f(x) = xe^{-x/4},$$

(a) Give the equation for the tangent line to $y = f(x)$ at $x = -4$. $\boxed{y = 2ex + 4e}$

(b) Compute the limits $\lim_{x\to 1} f(x)$ $\boxed{e^{-1/4}}$ and $\lim_{x\to\infty} f(x)$. $\boxed{0}$

(c) Find the critical point of $f(x)$. (There is only one.) $\boxed{x = 4}$ and $y = 4/e$

(d) Is the point from part (c) a local minimum, local maximum, or neither? $\boxed{\text{local max}}$

(e) Find the inflection point of $f(x)$. (There is only one.) $\boxed{x = 8}$ and $y = 8/e^2$

(f) Calculate the area of the domain $\{(x, y) : 0 \le x \le 4, 0 \le y \le f(x)\}$.

$$\int_0^4 f(x)\,\mathrm{d}x = \boxed{16 - \frac{32}{e}}$$