Math 1653
Analysis I

Wednesday 4 October 2023

Lecture instructor: dr Adam Abrams

Course format

Lecture (Wykład)

- Wednesdays 11:15 - 13:00 with dr Adam Abrams.

Problem session (Ćwiczenia) is either

- Tuesdays 18:55 - 20:35 with dr Adam Abrams or
- Thursdays 7:30 - 9:00 with dr Artur Rutkowski.

Lecture slides, tasks lists, and course policies are available at

theadamabrams.com/1653

Grading policy

The same grade is used for 1653W and 1653C.

- Six quizzes (5 points each), but the lowest score is ignored!
- Two exams (15 points each).
- Participation (5 points).

This makes $5 \times 5 + 15 + 15 + 5 = \mathbf{60}$ total possible points.

Points	[0, 30)	[30, 36)	[36, 42)	[42, 48)	[48, 54)	[54, 60]
Grade	2.0	3.0	3.5	4.0	4.5	5.0

Grading policy

The same grade is used for 1653W and 1653C.

Points	[0, 30)	[30, 36)	[36, 42)	[42, 48)	[48, 54)	[54, 60]
Grade	2.0	3.0	3.5	4.0	4.5	5.0

More than 4 unexcused absences after 6 Oct → course grade 2.0.

You can work together on task lists (which are not graded), but quizzes and exams are individual. All work can be checked in one-on-one meeting with either instructor.

- Cheating on a quiz → quiz grade 0.
- Cheating on exams → course grade 2.0.

Accessibility

Department of Accessibility and Support for People with Disabilities (DDO)

- Office: C-13 rooms 109 and 107
- Telephone: 71 320 43 20
- Website: https://ddo.pwr.edu.pl/
- Email: pomoc.n@pwr.edu.pl

If you need any kind of accommodation, please write me an email.
I am happy to help.

Topics

Limits

- Sequences
- Functions
- Continuity

Derivative calculations

- Power Rule
- Trig, log, exp
- Product Rule
- Chain Rule

Derivative applications

- Tangent lines
- Increasing and decreasing
- Concavity
- Min and max

Integrals

- Indefinite
- Definite
- Applications

Some students may already know some of these topics, but we will cover them all during this semester.

I gave this survey in yesterday's problem session.

Student ID: _ _ _ _ _ _

Name: ____________________

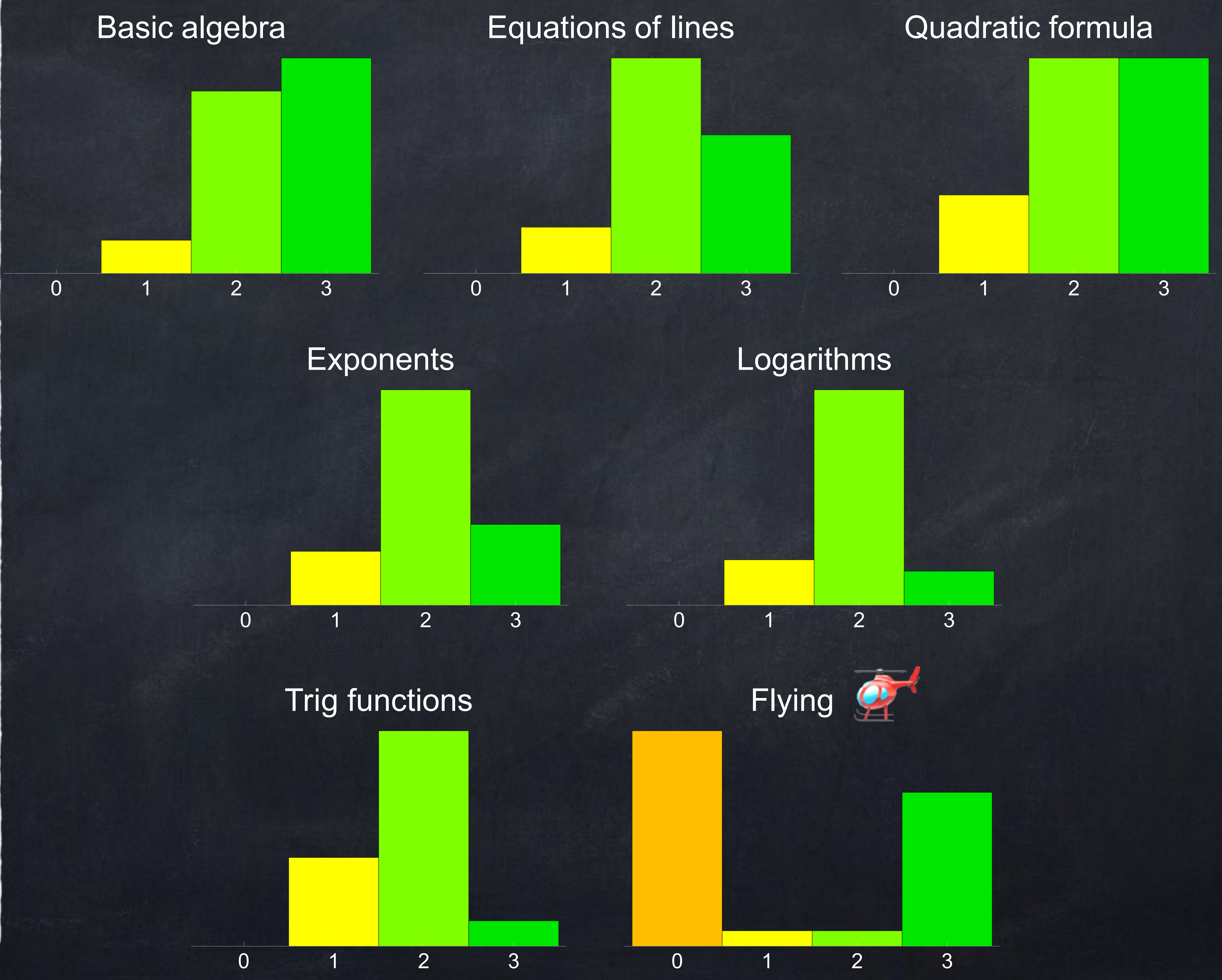
Preferred name: ____________

Favorite food:

Favorite book or movie or song:

How well do you know…

	Not at all	Poorly	Okay	Well
Basic algebra	0	1	2	3
Equations for lines	0	1	2	3
Quadratic formula	0	1	2	3
Exponents	0	1	2	3
Logarithms	0	1	2	3
Trig functions	0	1	2	3
How to fly a helicopter 🚁	0	1	2	3



Student ID: _ _ _ _ _ _ _

Name: ______________________

Preferred name: _____________

Favorite food:

Favorite book or movie or song:

How well do you know…

	Not at all	Poorly	Okay	Well
Basic algebra	0	1	2	3
Equations for lines	0	1	2	3
Quadratic formula	0	1	2	3
Exponents	0	1	2	3
Logarithms	0	1	2	3
Trig functions	0	1	2	3
How to fly a helicopter 🚁	0	1	2	3

1. The only way to become good at flying helicopters is to practice flying helicopters.

2. The only way to become good at doing mathematics is to practice doing mathematics.

Simply attending lectures and problem sessions is not enough!

Sequences

A **sequence** is an ordered list of numbers. It can be finite or infinite, but in this class we are interested in *infinite* sequences.

Examples:

- $(1, 52, 6, 6)$ is a finite sequence
- $(0, 2, 4, 6, 8, 10, 12, \ldots)$ is an infinite sequence
- $(1, 2, 4, 8, 16, 32, 64, 128, \ldots)$
- $(1, -1, 1, -1, 1, -1, \ldots)$
- $(1, 1, 3, 5, 8, 13, 21, 34, \ldots)$

We often write

$$a_n \qquad \text{(spoken as “A N” or “A sub N”)}$$

for the n^{th} term in a generic sequence. The number n is called the **index**.

Usually we start numbering with $n = 0$ or with $n = 1$.

$f(x)$ $g(x)$ $V(t)$

a_n b_n s_k

We often write

$$a_n \qquad \text{(spoken as "A N" or "A sub N")}$$

for the n^{th} term in a generic sequence. The number n is called the **index**. Usually we start numbering with $n = 0$ or with $n = 1$.

Here are four ways to describe the same sequence:

- $(1, 4, 9, 16, 25, 36, 49, \ldots)$
- $a_1 = 1, a_2 = 4, a_3 = 9,$ etc.
- $a_n = n^2$ for $n \geq 1$.

"A sub three equals nine"
or "A three equals nine"

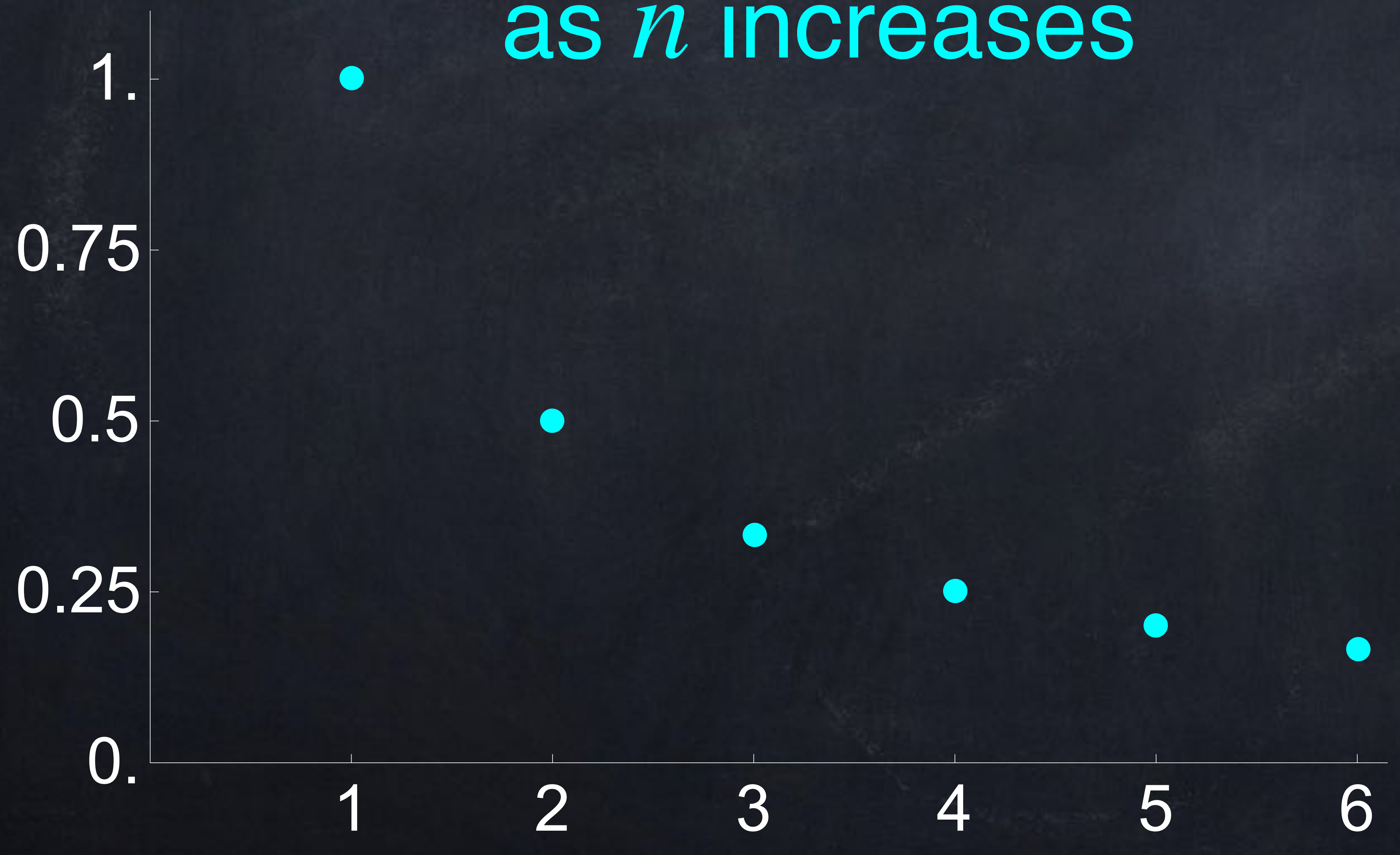
Topics that are *not* part of this course:

- definition of “arithmetic” sequence,
- definition of “geometric” sequence,
- recursive formulas for sequences (e.g., $a_n = a_{n-1} + 2n - 1$).

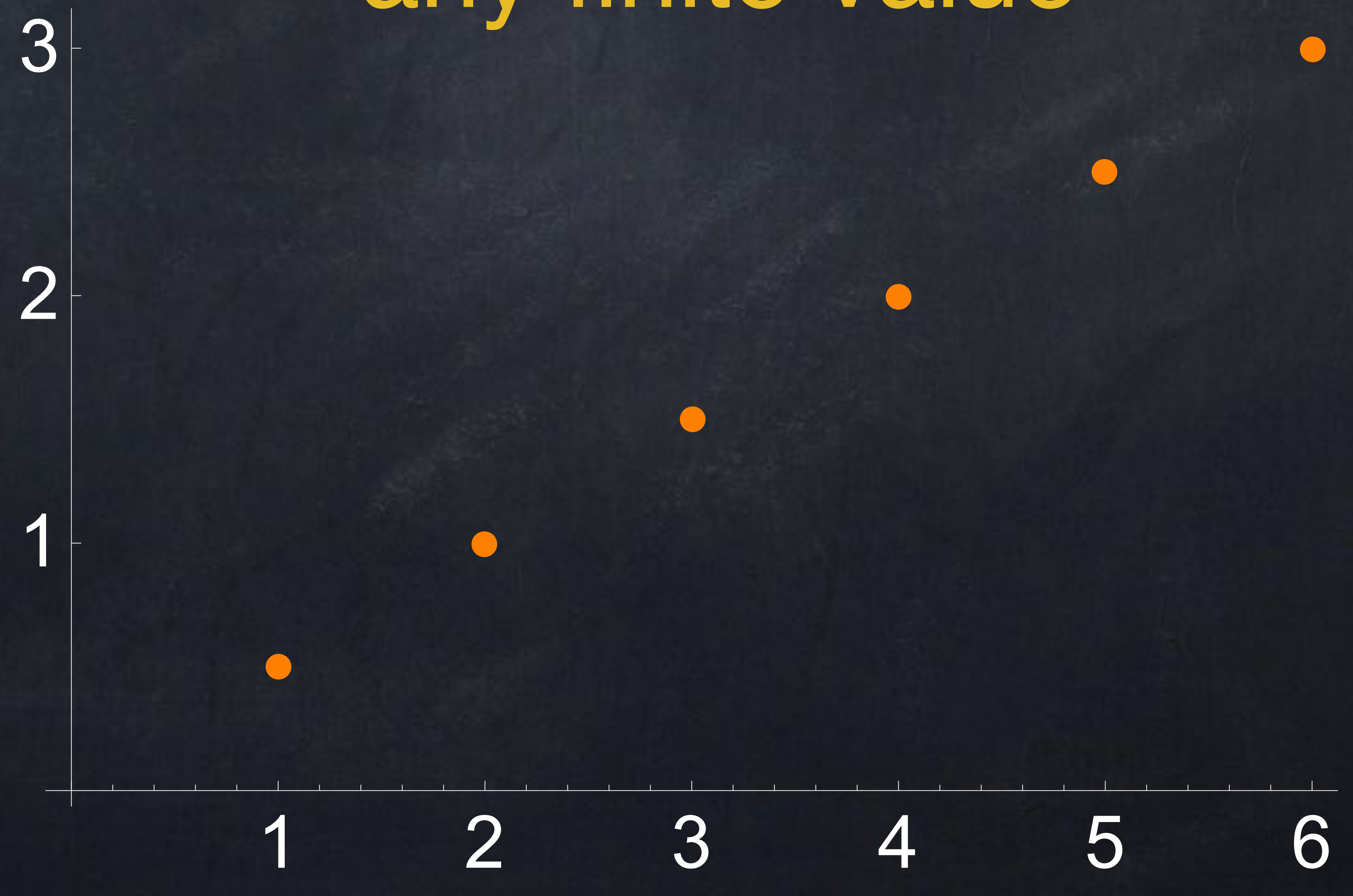
Convergence

An infinite sequence goes on forever, but sometimes the values a_n for very large n get closer and closer to a single value.

$a_n = \frac{1}{n}$ gets close to 0 as n increases



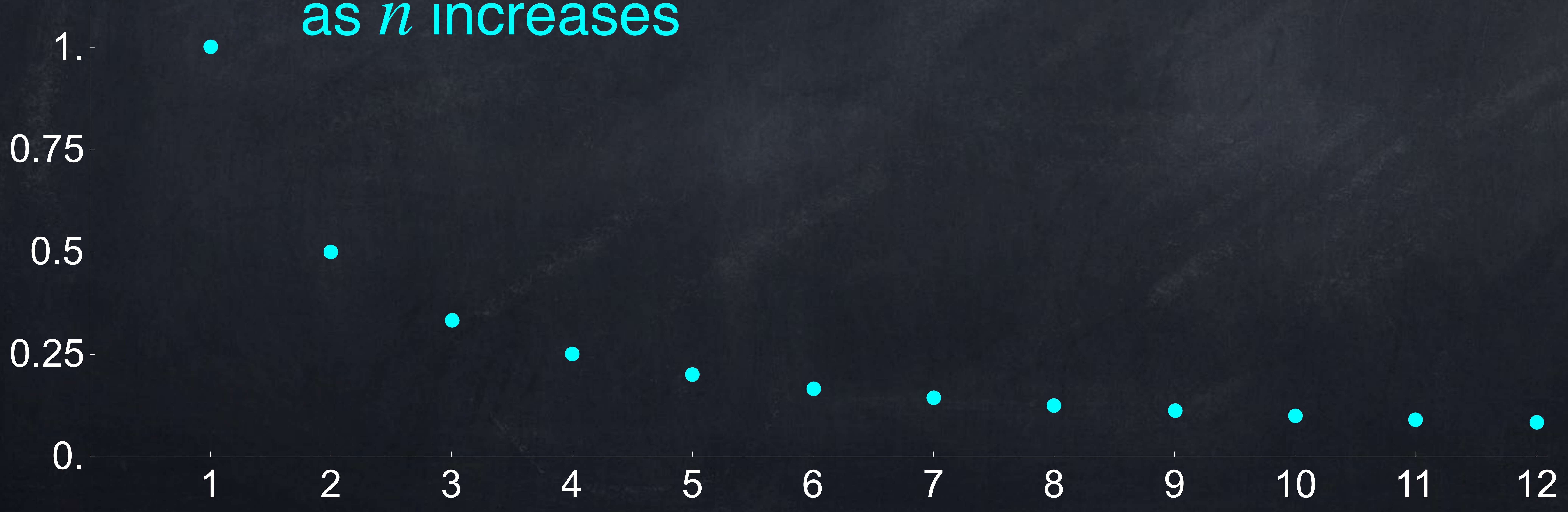
$a_n = \frac{n}{2}$ does not approach any finite value



Convergence

An infinite sequence goes on forever, but sometimes the values a_n for very large n get closer and closer to a single value.

$a_n = \frac{1}{n}$ gets close to 0 as n increases



Convergence

An infinite sequence goes on forever, but sometimes the values a_n for very large n get closer and closer to a single value.

$a_n = \dfrac{5n - 4}{n}$ gets close to 5 as n increases

If the values a_n get closer and closer to some real number L as n gets very large, then we can say

- "a_n converges" or
- "a_n converges to L" or
- "the limit of a_n is L" or
- "the limit of a_n is L" or
- "the limit of a_n as n goes to infinity is L" or
- "the limit as n goes to infinity of a_n"is L,

and we can write

$$\lim_{n\to\infty} a_n = L.$$

Calculating Limits

This is *official* definition of a limit:

If L is a real number, then

$$\lim_{n \to \infty} a_n = L$$

means that for any $\varepsilon > 0$ there exists an N such that

$$L - \varepsilon < a_{N+1} < L + \varepsilon$$
$$L - \varepsilon < a_{N+2} < L + \varepsilon$$
$$L - \varepsilon < a_{N+3} < L + \varepsilon$$
$$L - \varepsilon < a_{N+4} < L + \varepsilon$$
$$\vdots$$

Calculating Limits

This is *official* definition of a limit:

If L is a real number, then

$$\lim_{n\to\infty} a_n = L$$

means that for any $\varepsilon > 0$ there exists an N such that

$$L - \varepsilon < a_n < L + \varepsilon$$

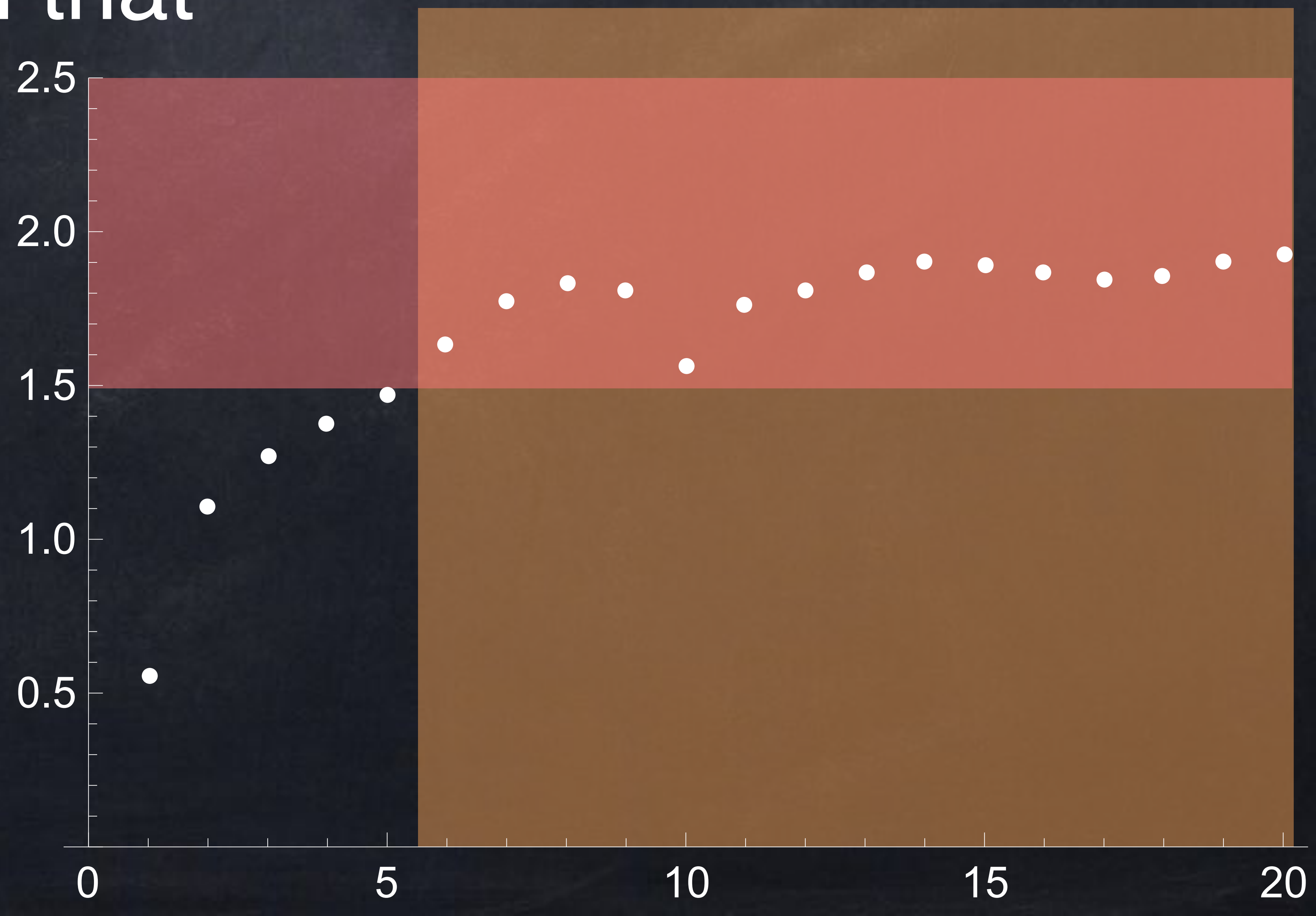
for all $n > N$.

It's common to write

$$a_n - L \quad < \varepsilon$$

instead.

For $\varepsilon = \frac{1}{2}$, we can use $N = 5$.



Calculating Limits

This is *official* definition of a limit:

If L is a real number, then

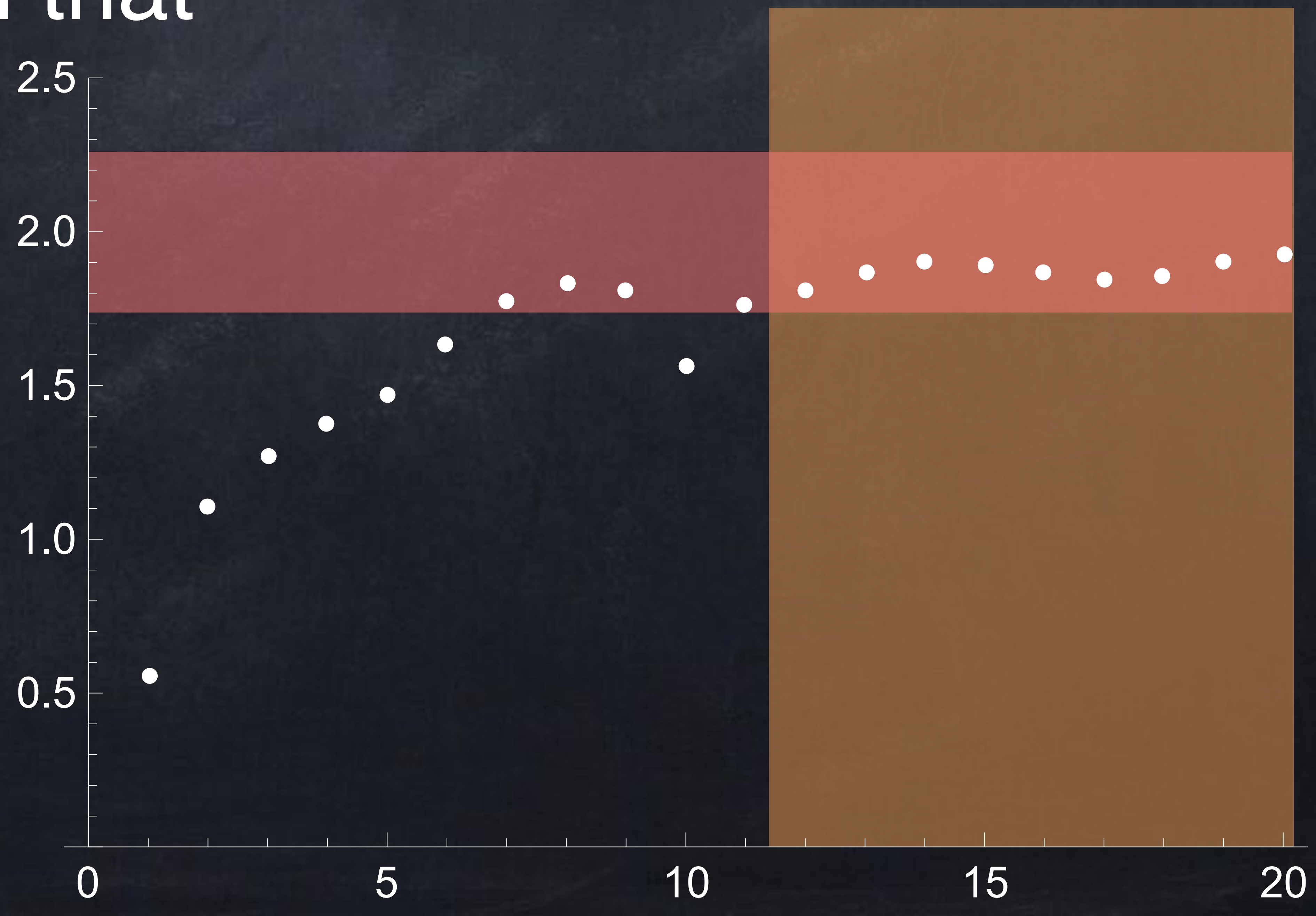
$$\lim_{n\to\infty} a_n = L$$

means that for any $\varepsilon > 0$ there exists an N such that

$$L - \varepsilon < a_n < L + \varepsilon$$

for all $n > N$.

For $\varepsilon = \frac{1}{4}$, we can use $N = 11$.



Calculating Limits

This is *official* definition of a limit:

If L is a real number, then

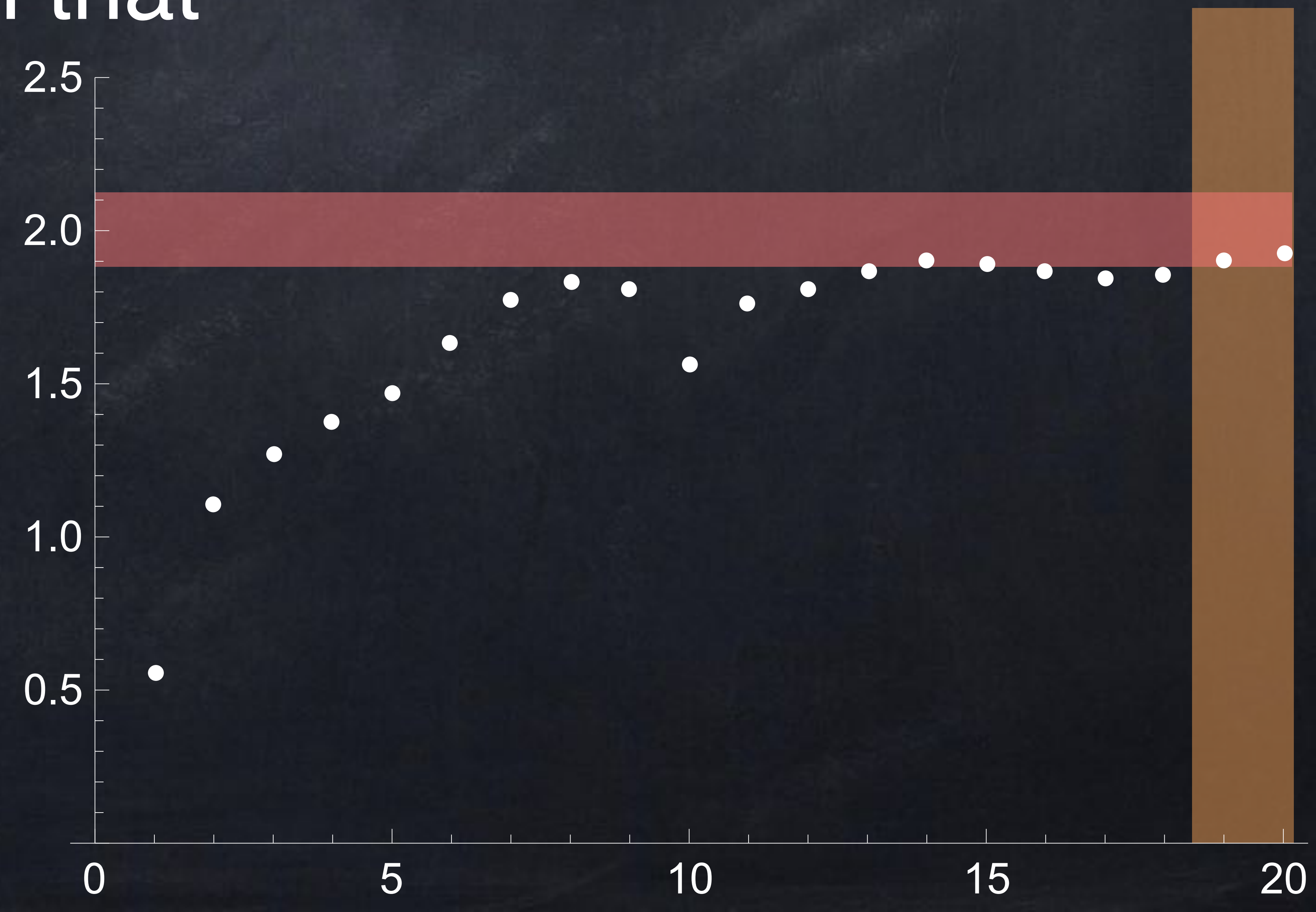
$$\lim_{n\to\infty} a_n = L$$

For ε = 0.1, we can use N = 19.

means that for any $\varepsilon > 0$ there exists an N such that

$$L - \varepsilon < a_n < L + \varepsilon$$

for all $n > N$.



What does $\lim_{n\to\infty} \frac{n}{n+5} = 1$ mean?

- It's possible to guarantee $0.8 < \frac{n}{n+5} < 1.2$ for all $n > N$.

  (To get this, we need N to be at least 20.)

- It's possible to guarantee $0.9 < \frac{n}{n+5} < 1.1$ for all $n > N$.

  (To get this, we need N to be at least 45.)

- It's possible to guarantee $0.9999 < \frac{n}{n+5} < 1.0001$ for all $n > N$.

  (To get this, we need N to be at least 49995.)

Formal proof that $\lim_{n\to\infty} \frac{n}{n+5} = 1$:

Given ε, we want to find an N that guarantees $1 - \varepsilon < \frac{n}{n+5} < 1 + \varepsilon$.

$$\frac{n}{n+5} > 1 - \varepsilon$$

$$n > (1 - \varepsilon)(n + 5)$$

$$n > n - n\varepsilon + 5 - 5\varepsilon$$

$$0 > -n\varepsilon + 5 - 5\varepsilon$$

$$n > \frac{5 - 5\varepsilon}{\varepsilon}$$

So we can use $N = \frac{5 - 5\varepsilon}{\varepsilon}$ rounded up.

Calculating Limits

Instead of using the "N, ε definition", for this class it is usually enough to think carefully about what values occur for large n.

Example: What is $\lim_{n\to\infty} \frac{n+3}{2n}$?

- For small values of n we get $a_1 = 2$, $a_2 = \frac{5}{4}$, etc., but those values are not important when we talk about the limit.
- We care more about $a_{1000} = \frac{1003}{2000} = 0.5015$ and $a_{1000000} = \frac{1000003}{2000000} = 0.5000015$.
- As n gets larger and larger, a_n will get even closer to the number 0.5.

Calculating Limits

Instead of using the "N, ε definition", for this class it is usually enough to think carefully about what values occur for large n.

Example: What is $\lim_{n\to\infty} \frac{n+3}{2n}$?

- We don't have to plug in only powers of 10. It's also true that

$$a_{874657} = \frac{874660}{1749314} \approx 0.50000171495$$

for this sequence.

Answer: $\lim_{n\to\infty} \frac{n+3}{2n} = \frac{1}{2}$.

What does $\lim_{n\to\infty} \dfrac{n}{n+5} = 1$ mean? (Again.)

- *Formally:* for any $\varepsilon > 0$,

  if $n > \dfrac{5 - 5\varepsilon}{\varepsilon}$ then $1 - \varepsilon < \dfrac{n}{n+5} < 1 + \varepsilon$.

- *Informally:*

  if n is very big then $\dfrac{n}{n+5}$ is very close to 1.

More examples:

- $$\lim_{n\to\infty} \frac{n^3 - 4n + 1}{6n^3 + 8} = ?$$

  1/6

  Answer: ~~0~~

- $$\lim_{n\to\infty} \frac{\sin(2n)}{n^5} = ?$$

  Answer: 0

- $$\lim_{n\to\infty} (-1)^n = ?$$

  Answer: this limit does not exist.

Limit rules

If the limits all exist and are finite, then

- $\lim\limits_{n\to\infty}\left(a_n + b_n\right) = \left(\lim\limits_{n\to\infty} a_n\right) + \left(\lim\limits_{n\to\infty} b_n\right),$
- $\lim\limits_{n\to\infty}\left(a_n \cdot b_n\right) = \left(\lim\limits_{n\to\infty} a_n\right)\left(\lim\limits_{n\to\infty} b_n\right),$
- $\lim\limits_{n\to\infty}\dfrac{a_n}{b_n} = \dfrac{\lim_{n\to\infty} a_n}{\lim_{x\to a} b_n}$ if $\lim\limits_{n\to\infty} b_n \neq 0,$
- $\lim\limits_{n\to\infty} a_n^p = \left(\lim\limits_{n\to\infty} a_n\right)^p$ for any real number p.

with $b_n = c$ constant

- $\lim\limits_{n\to\infty}\left(c \cdot a_n\right) = c \cdot \left(\lim\limits_{n\to\infty} a_n\right)$

Limit rules

It will often be useful to know the limit of r^n (note n is a power, not index).

- If $-1 < r < 1$ then $\lim_{n\to\infty} r^n = 0$.
- If $r = 1$ then $\lim_{n\to\infty} r^n = 1$.
- If $r \leq -1$ then $\lim_{n\to\infty} r^n$ does not exist.
- If $r > 1$ then $\lim_{n\to\infty} r^n = \infty$.

This does not mean "$\infty - \varepsilon < a_n < \infty + \varepsilon$" for all $n > N$ (that's nonsense).

Infinite Limits

Reminder: we write $\lim_{n\to\infty} a_n = L$ if for every $\varepsilon > 0$ there exists N such that

$$L - \varepsilon < a_n < L + \varepsilon \quad \text{for all } n > N.$$

The sequences $b_n = 2^n$ and $c_n = (-1)^n$ both do not have real limits, but for very different reasons.

We will write

$$\text{“}\lim_{n\to\infty} 2^n = \infty\text{”},$$

but this does *not* use the definition above.

Infinite Limits

Reminder: we write $\lim_{n\to\infty} a_n = L$ if for every $\varepsilon > 0$ there exists N such that

$$L - \varepsilon < a_n < L + \varepsilon \quad \text{for all } n > N.$$

New definitions:

- We write $\lim_{n\to\infty} a_n = \infty$ if for every $M > 0$ there exists N such that

$$a_n > M \quad \text{for all } n > N.$$

- We write $\lim_{n\to\infty} a_n = -\infty$ if for every $M > 0$ there exists N such that

$$a_n < -M \quad \text{for all } n > N.$$

Limit rules

When we have a ratio of two *polynomials*, the limit

$$\lim_{n\to\infty} \frac{An^d + \cdots}{Bn^e + \cdots}$$

can be found very quickly. (Here "$\cdots$" are terms with smaller powers of n).

- If $d < e$ then the limit is 0.
- If $d = e$ then the limit is $\frac{A}{B}$.
- If $d > e$ then
  - the limit is ∞ if $\frac{A}{B} > 0$.
  - the limit is $-\infty$ if $\frac{A}{B} < 0$.

Task: calculate each of the following limits, if they exist.

- $$\lim_{n\to\infty} \left(\sqrt{n^2+2} - n\right) = 0$$
- $$\lim_{n\to\infty} \frac{5n^3 - 2n + 6}{2n^3 + n^2 + 180n - 1} = \frac{5}{2}$$
- $$\lim_{n\to\infty} \frac{5n^2 - 2n + 6}{2n^3 + n^2 + 180n - 1} = 0$$
- $$\lim_{n\to\infty} (1+n)^n = \infty$$
- $$\lim_{n\to\infty} (1+n)^{1/n} = 1$$
- $$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$$

≈ 2.718

This last one is good to memorize.