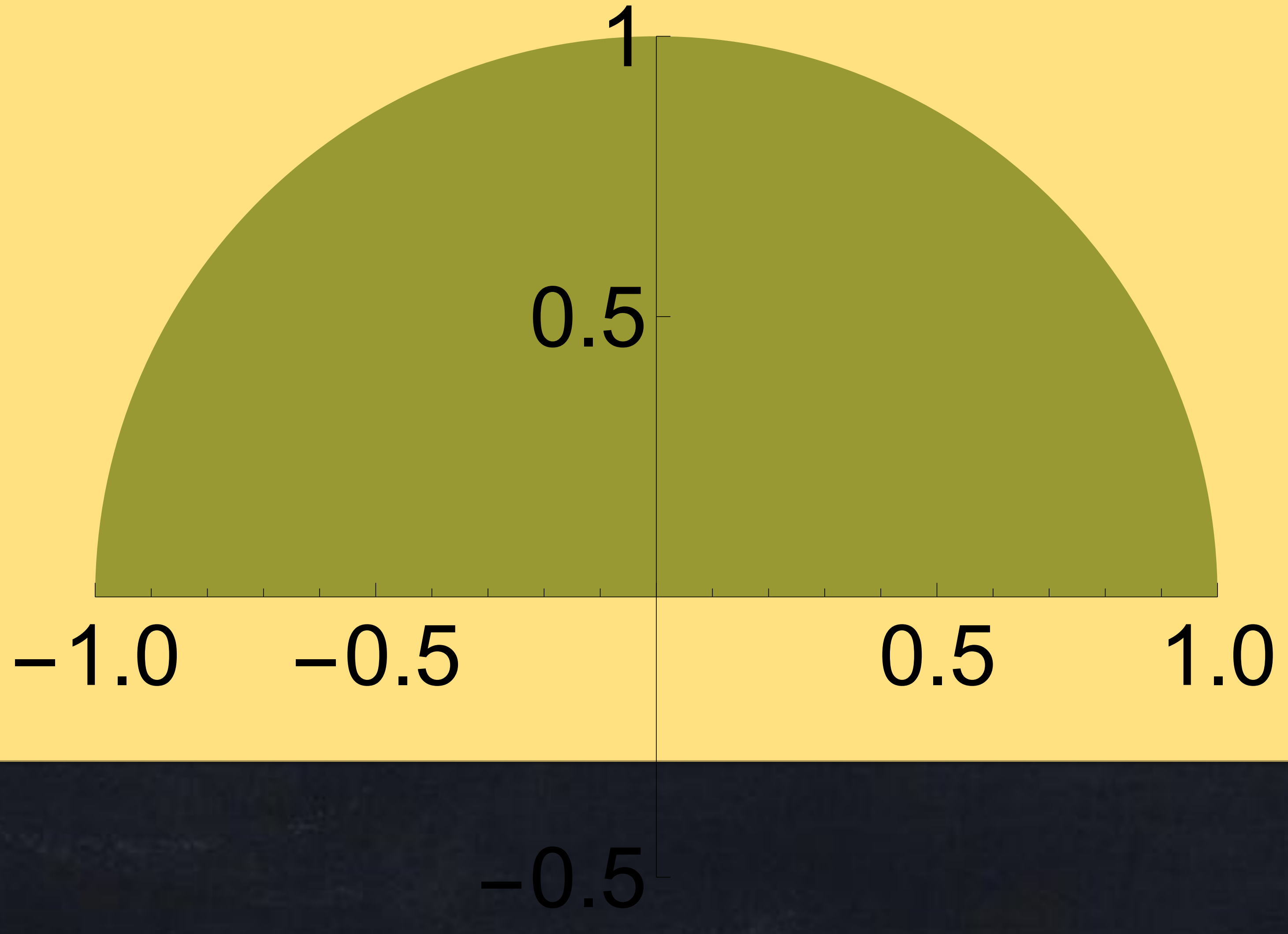
Analysis 1

17 January 2024

Warm-up: Write an integral that calculates this area:



"substitution"

Last time

Example: using $u = x^3$ and $\mathrm{d}u = 3x^2\mathrm{d}x$, we can rewrite $\int x^2 \sin(x^3)\,\mathrm{d}x = \int \sin(u)\frac{1}{3}\mathrm{d}u$, which is $\frac{-1}{3}\cos(u) + C = \frac{-1}{3}\cos(x^3) + C$.

This method is useful for some integrals, but not for others.

"parts"

New

This is another method to rewrite an integral into one that may be easier to do.

It is especially useful for *some* integrals with trig and exponential functions.

More substitutions

Fill in the part that is missing.

$u = x^5 - 1$

$du = ?$

$v = ?$

$dv = x\,dx$

$s = e^{-8x}$

$ds = ?$

$w = ?$

$dw = e^{8x}\,dx$

More substitutions

Fill in the part that is missing.

$u = x^5 - 1$

$du = 5x^4 dx$

$v = \frac{1}{2}x^2$

$dv = x\, dx$

$s = e^{-8x}$

$ds = -8e^{-8x} dx$

$w = \frac{1}{8}e^{8x}$

$dw = e^{8x}\, dx$

(Hallway activity with papers)

Products

We know that $\frac{\mathrm{d}}{\mathrm{d}x}(f \cdot g)$ is NOT $\frac{\mathrm{d}f}{\mathrm{d}x} \cdot \frac{\mathrm{d}g}{\mathrm{d}x}$. The product rule says that $(fg)' = fg' + f'g$ instead. Similarly, $\int (f \cdot g)\mathrm{d}x$ is NOT $\left(\int f \mathrm{d}x\right) \cdot \left(\int g \mathrm{d}x\right)$.

Question: How can we do $\int \_\_\_\_ \cdot \_\_\_\_ \mathrm{d}x$?

- As an official formula, **substitution** is $\int f(u(x)) \cdot u'(x)\, \mathrm{d}x = \int f(u) \mathrm{d}u$.
- **Integration by parts** can handle many other products.

Integration by parts

Substitution works for some integrals but not for others. Here is another formula that works for some (more difficult) integrals:

Product rule but with integrals:

$$\int f g' \mathrm{d}x + \int f' g \,\mathrm{d}x = fg$$

$v = f$
$dv = f'dx$

$u = g$
$du = g'dx$

$$\int v \,\mathrm{d}u + \int u \,\mathrm{d}v = u\,v$$

$$\int u \,\mathrm{d}v = u\,v - \int v \,\mathrm{d}u$$

ultraviolet voodoo

might help you remember this formula

Example 1: $\int \underbrace{2x}_{u}\ \underbrace{\cos(3x)\,\mathrm{d}x}_{dv}$

Parts:

$$\int u\,\mathrm{d}v + \int v\,\mathrm{d}u = u\,v$$

$$\int (2x)(\cos(3x)dx) = (2x)\left(\frac{1}{3}\sin(3x)\right) - \int \left(\frac{1}{3}\sin(3x)\right)(2dx)$$

$$= \frac{2}{3}x\sin(3x) - \int \frac{2}{3}\sin(3x)dx$$

$$= \boxed{\frac{2}{3}x\sin(3x) - \frac{2}{9}\cos(3x) + C}$$

How do you choose good u and $\mathrm{d}v$?

- Generally, u should have a derivative that is similar or less complicated than u.

$$\begin{array}{cccc} u = 8x+6 & 3x^2 & \sin(5x) & e^{9x} \\ \downarrow & \downarrow & \downarrow & \downarrow \\ du = 8\,dx & 6x & 5\cos(5x) & 9e^{9x} \end{array}$$

- Generally, $\frac{\mathrm{d}v}{\mathrm{d}x}$ should have an *anti*-derivative that is similar to $\frac{\mathrm{d}v}{\mathrm{d}x}$. Therefore trig and exponential functions are often good choices for $\frac{\mathrm{d}v}{\mathrm{d}x}$.

$$\begin{array}{cc} v = -\cos(x) & \frac{1}{2}e^{2x} \\ \uparrow & \uparrow \\ dv = \sin(x)\,dx & e^{2x} \end{array}$$

Task 2: $\int 4\, e^{2x}\, x^2\, \mathrm{d}x = ? + C$

Parts:

$$\int u\, \mathrm{d}v = u\, v - \int v\, \mathrm{d}u$$

This requires "parts" <u>twice</u>.

Answer after simplifying: $(2x^2 - 2x + 1)e^{2x} + C$

Task 3: Calculate $I = \int_0^{10} 4\,x^2\,e^{2x}\mathrm{d}x$.

$I = 181e^{20} - 1$

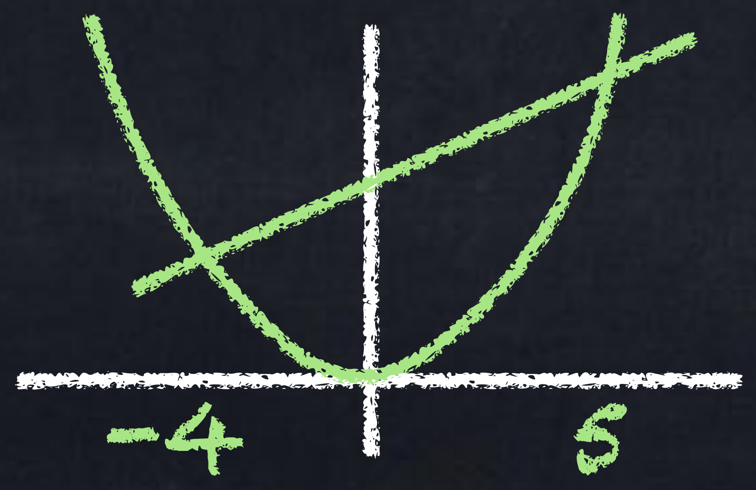
Calculation vs application

Derivatives:

- Find the derivative of $f(x) = \cos(x^3)\sqrt{\sin(x)}$. Product rule, chain rule.
- Find the local maximum of $f(x) = e^{-x/3}\sqrt{x}$. $f'(x) = 0$, then First Deriv. Test or Second Deriv. Test

Integrals:

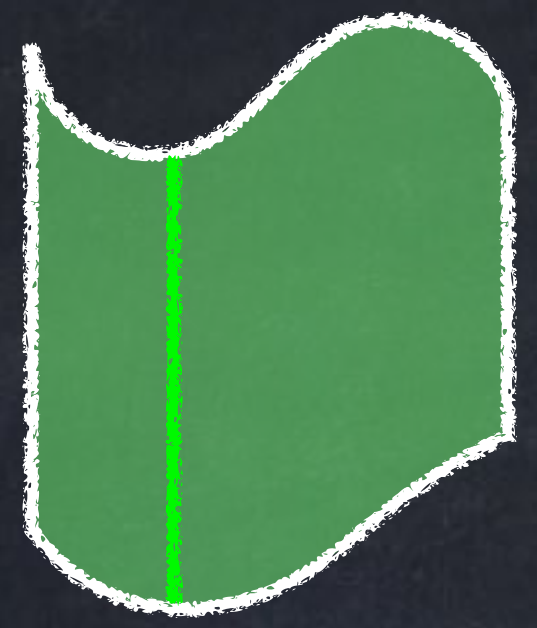
- Find $\displaystyle\int 5x\sin(3x)\,\mathrm{d}x$. Parts
- Find the area of the region between $y = x^2$ and $y = x + 20$.



$$\int_{-4}^{5} \left((x+20) - x^2\right) dx$$

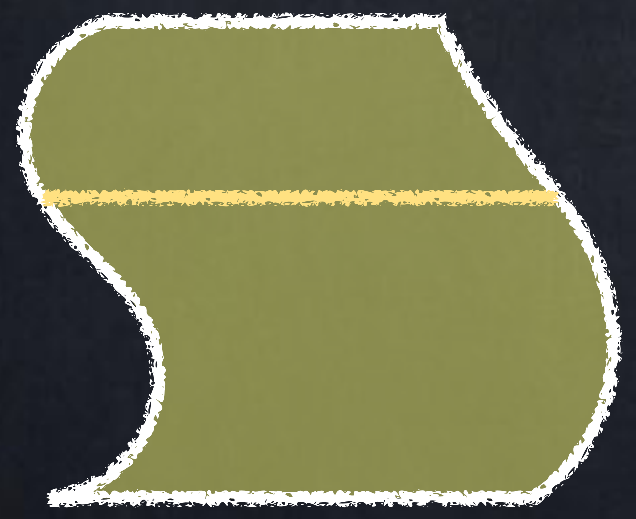
Area is integral of height or width

The area of a shape with $a \leq x \leq b$ and with curves on the top and bottom is



$$\int_a^b \big(\underline{\mathrm{Top}(x) - \mathrm{Bottom}(x)}\big)\,\mathrm{d}x.$$

The area of a shape with $c \leq y \leq d$ and with curves on the left and right is



$$\int_c^d \big(\underline{\mathrm{Right}(y) - \mathrm{Left}(y)}\big)\,\mathrm{d}y.$$

For some shapes, both methods are possible!

Volume as an integral

We have seen that

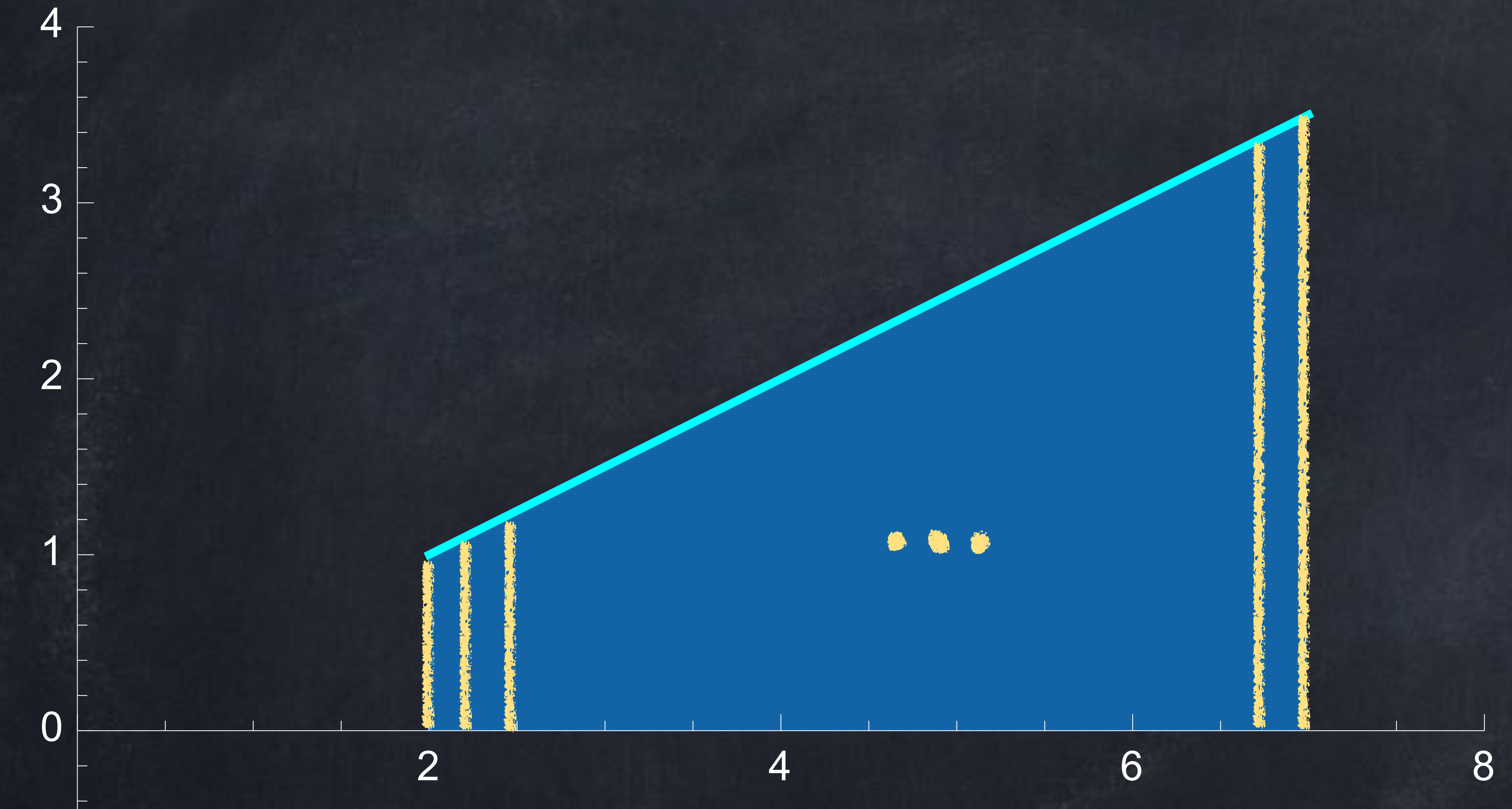
$$\text{Area} = \int (\text{height})\,\mathrm{d}x.$$

Increasing from 2D to 3D, we have

$$\text{Volume} = \int (\text{area of cross-section})\,\mathrm{d}x.$$

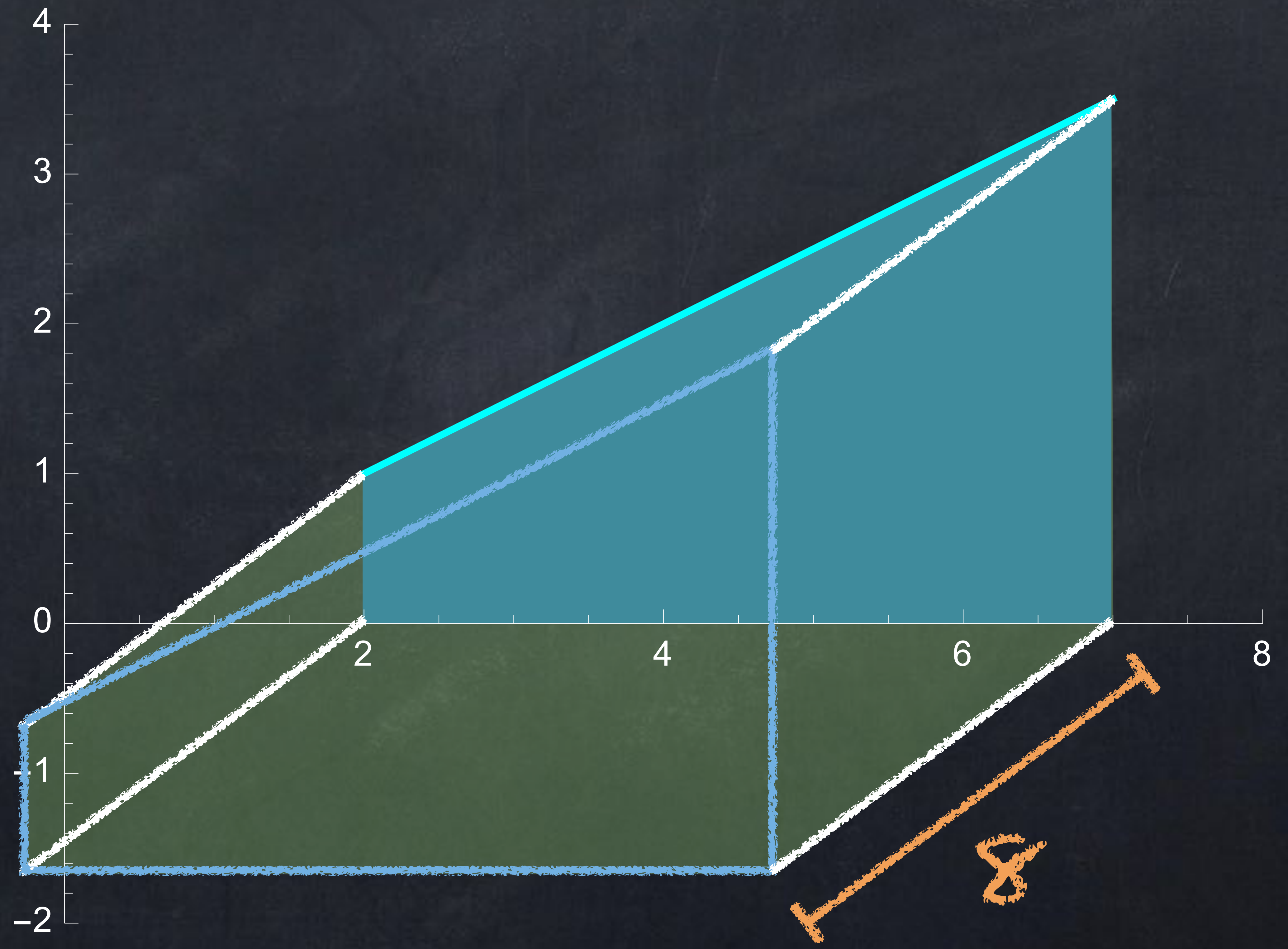
This is useful for “solids of revolution”.

What is the area of the region with $2 \leq x \leq 7$ bounded by $y = \frac{1}{2}x$ and the x-axis?



$$A = \int_2^7 \frac{x}{2}\,dx = \frac{45}{4}$$

What is the volume of this 3D shape?



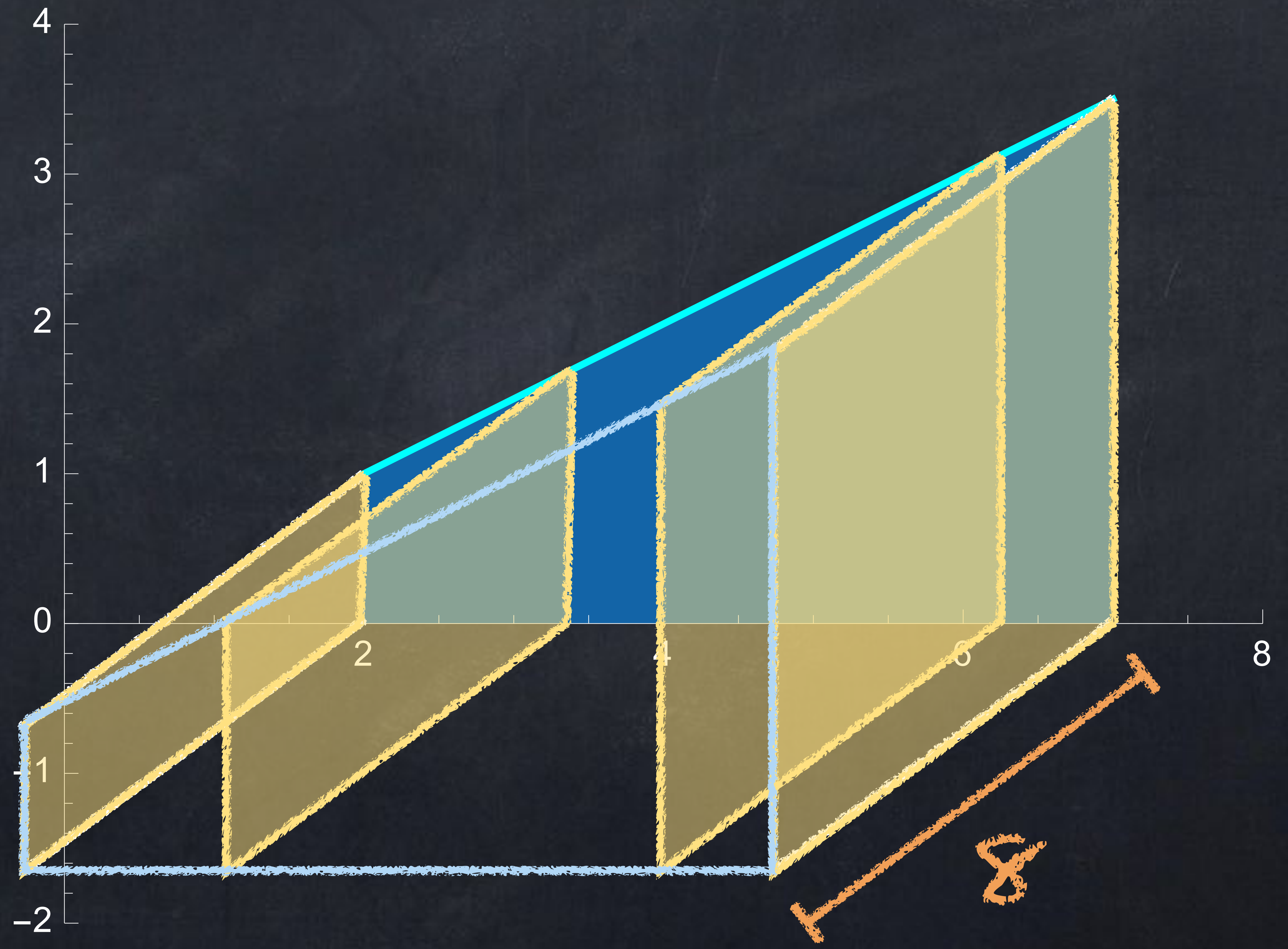
$$\text{Volume} = \int_2^7 \text{Area}\, dx$$

$$= \int_2^7 \frac{x}{2} \cdot 8\, dx$$

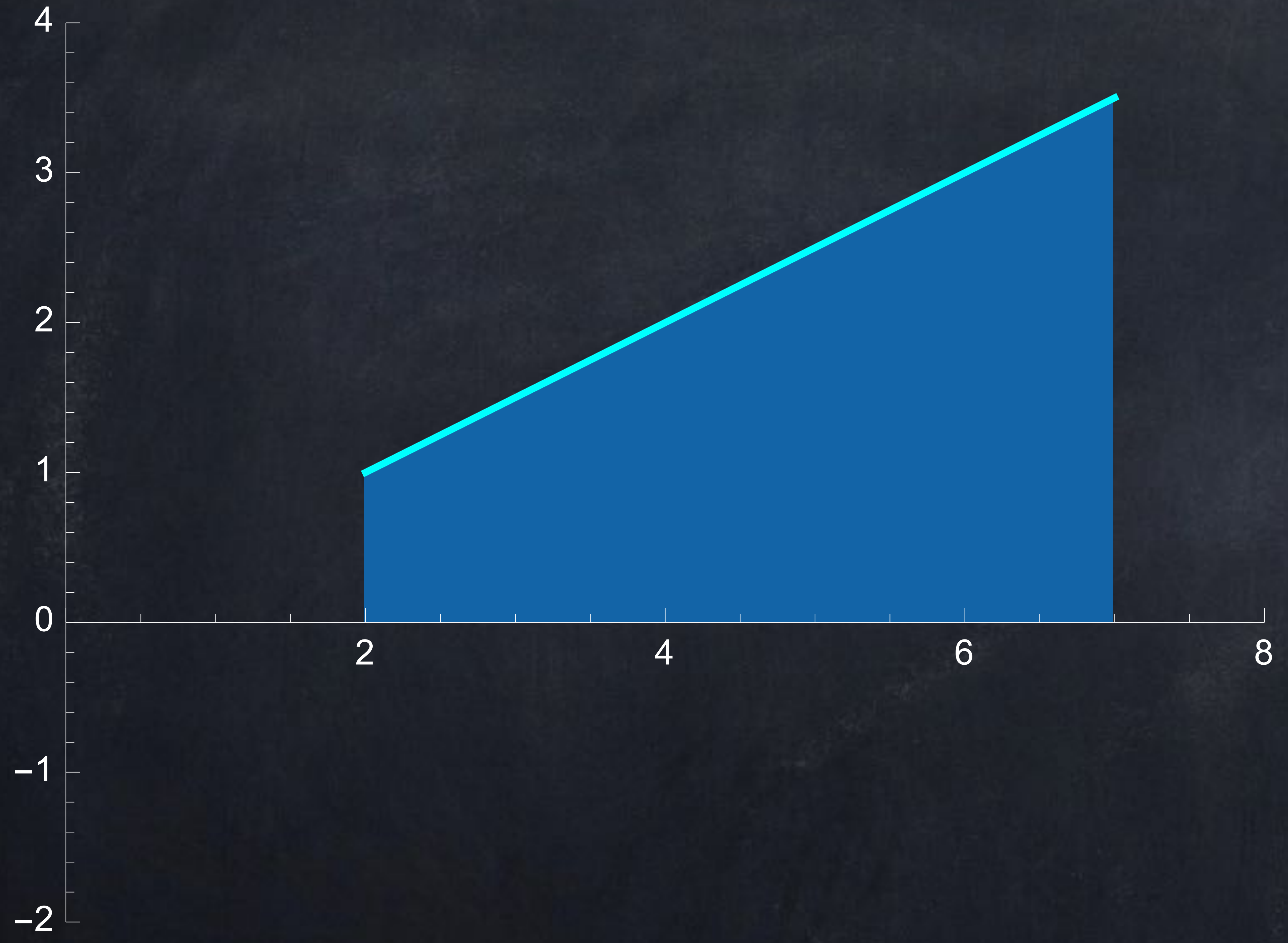
$$= 90$$

In this example, we can also just say $V = \frac{45}{4} \times 8 = 90$, but the idea of looking at cross-sections will be helpful for other shapes.

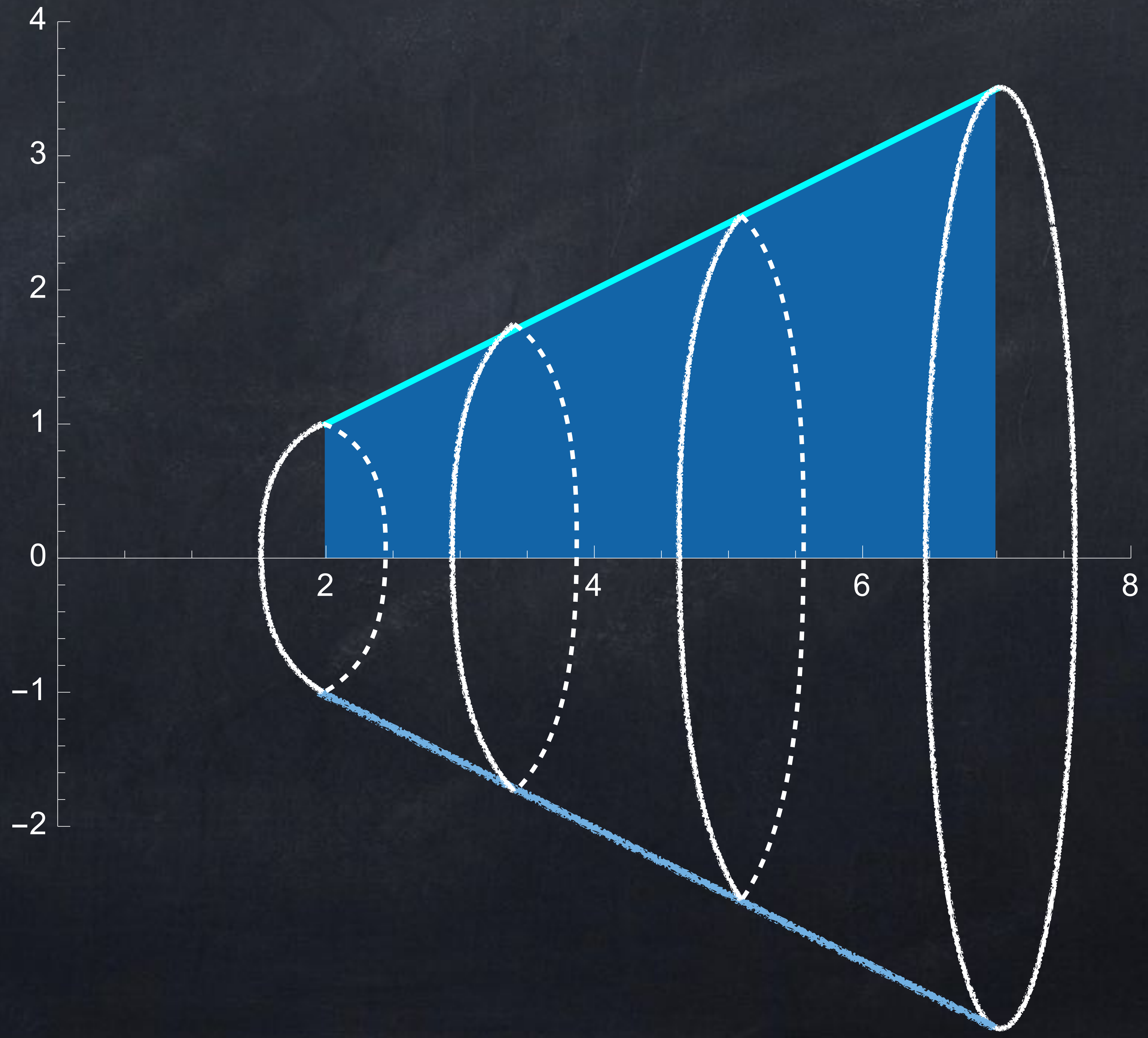
What is the volume of this 3D shape?



Start with the region with $2 \leq x \leq 7$ bounded by $y = \frac{1}{2}x$ and the x-axis…



… and rotate (or spin, or *revolve*) this shape around the x-axis.



$$\text{Volume} = \int_2^7 \text{Area}\, dx$$

$$= \int_2^7 \pi (\text{radius})^2\, dx$$

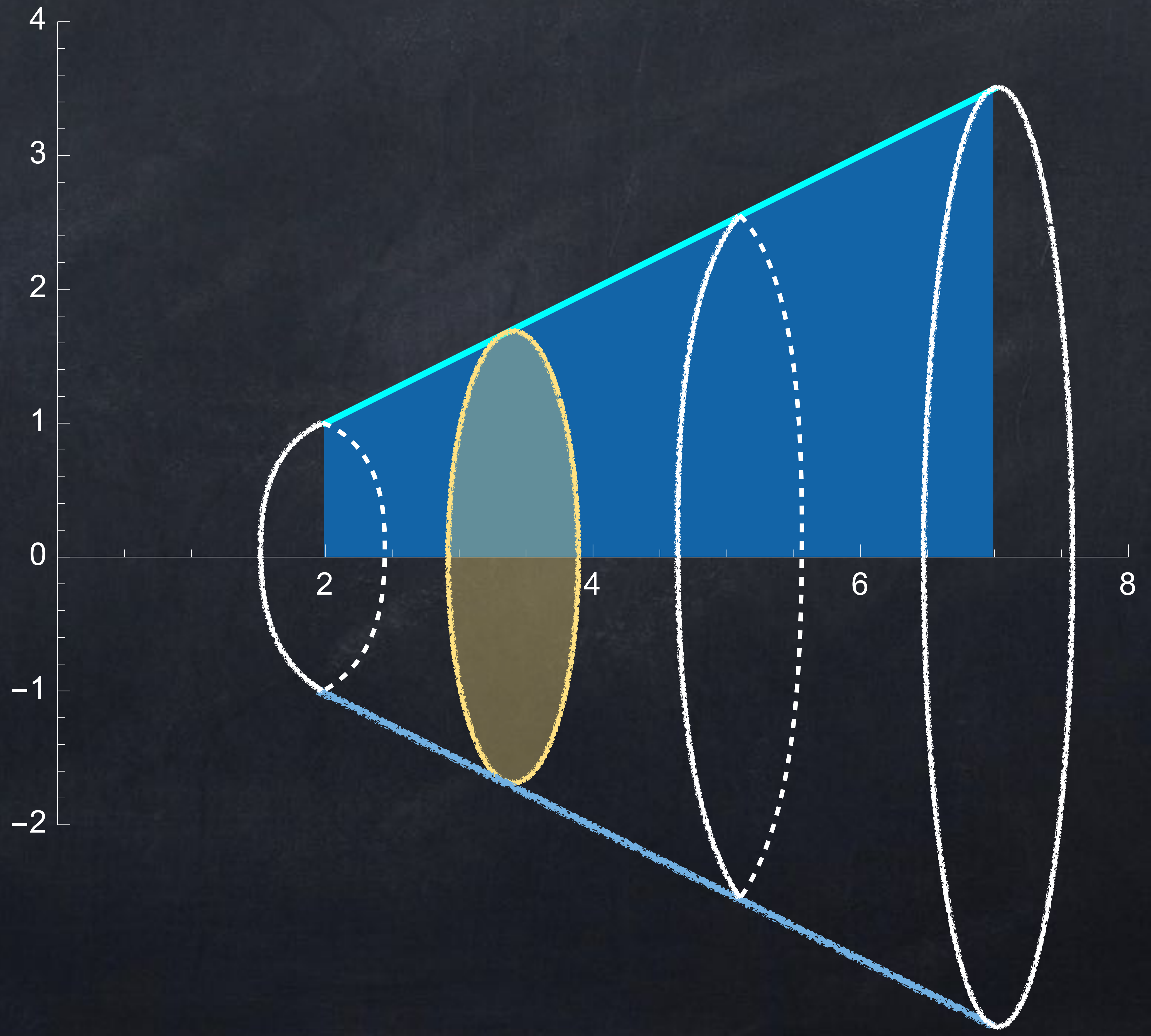
$$= \int_2^7 \pi \left(\frac{x}{2}\right)^2 dx$$

$$= \frac{(7)^3}{12}\pi - \frac{(2)^3}{12}\pi$$

$$= \frac{355}{12}\pi$$

… and rotate (or spin, or *revolve*) this shape around the x-axis.

What is the volume?

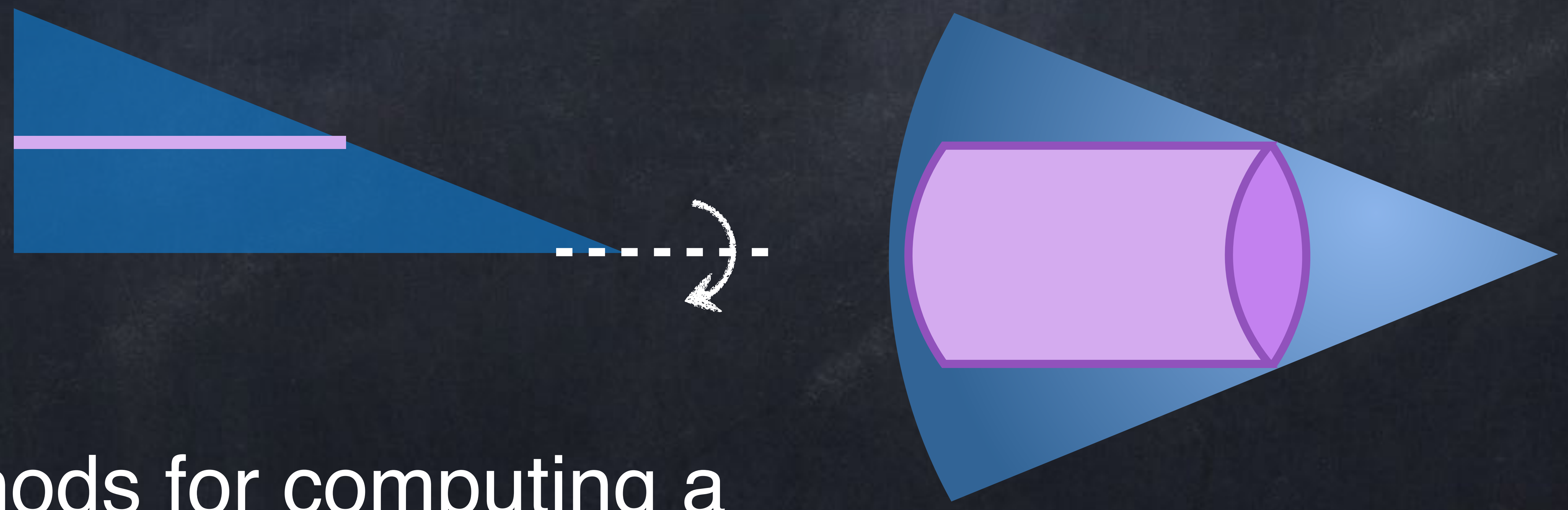


Area is integral of height or width

The area of a 2D region can be computed as either

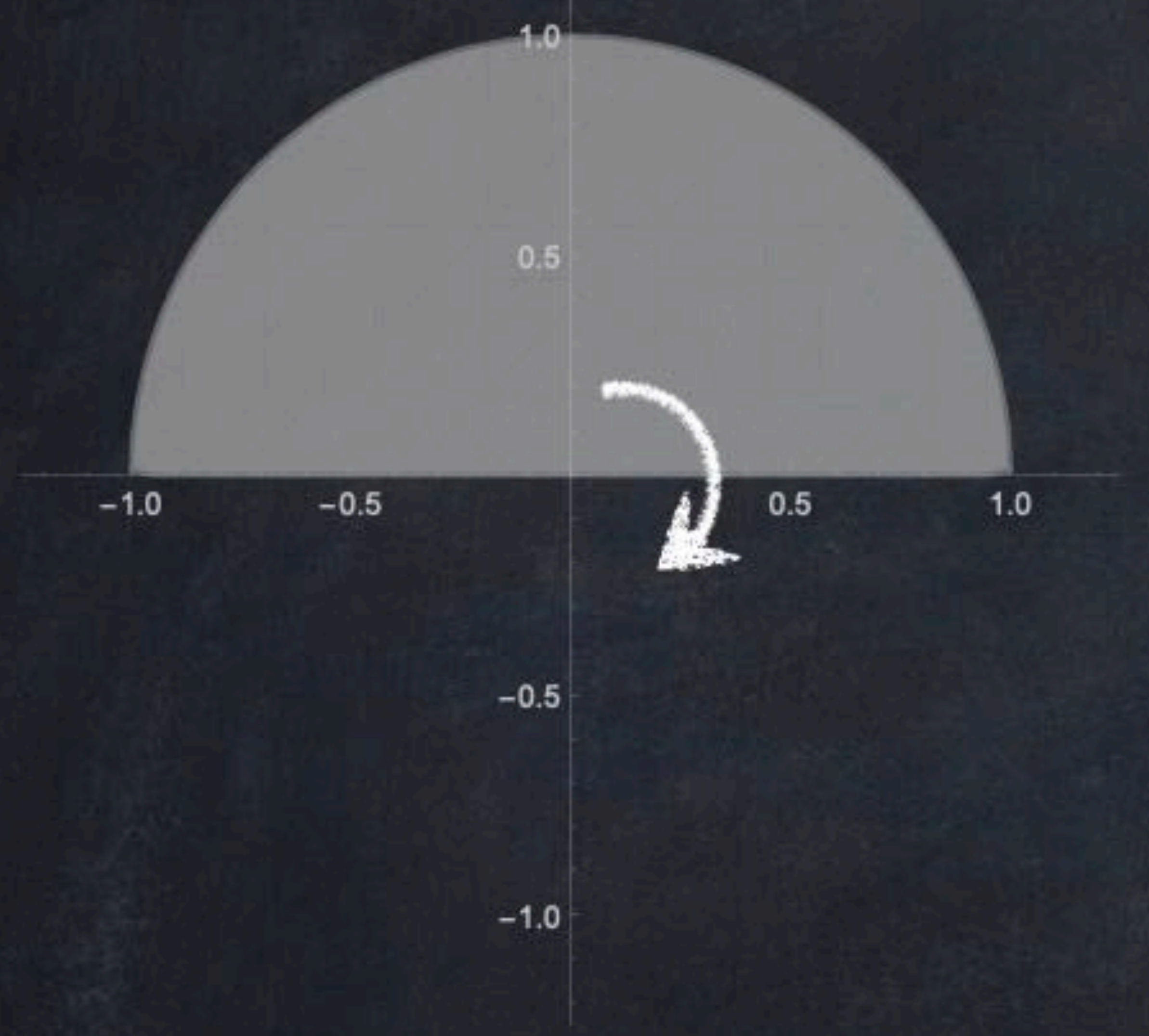
- $$\text{Area} = \int_L^R h(x)\,\mathrm{d}x$$

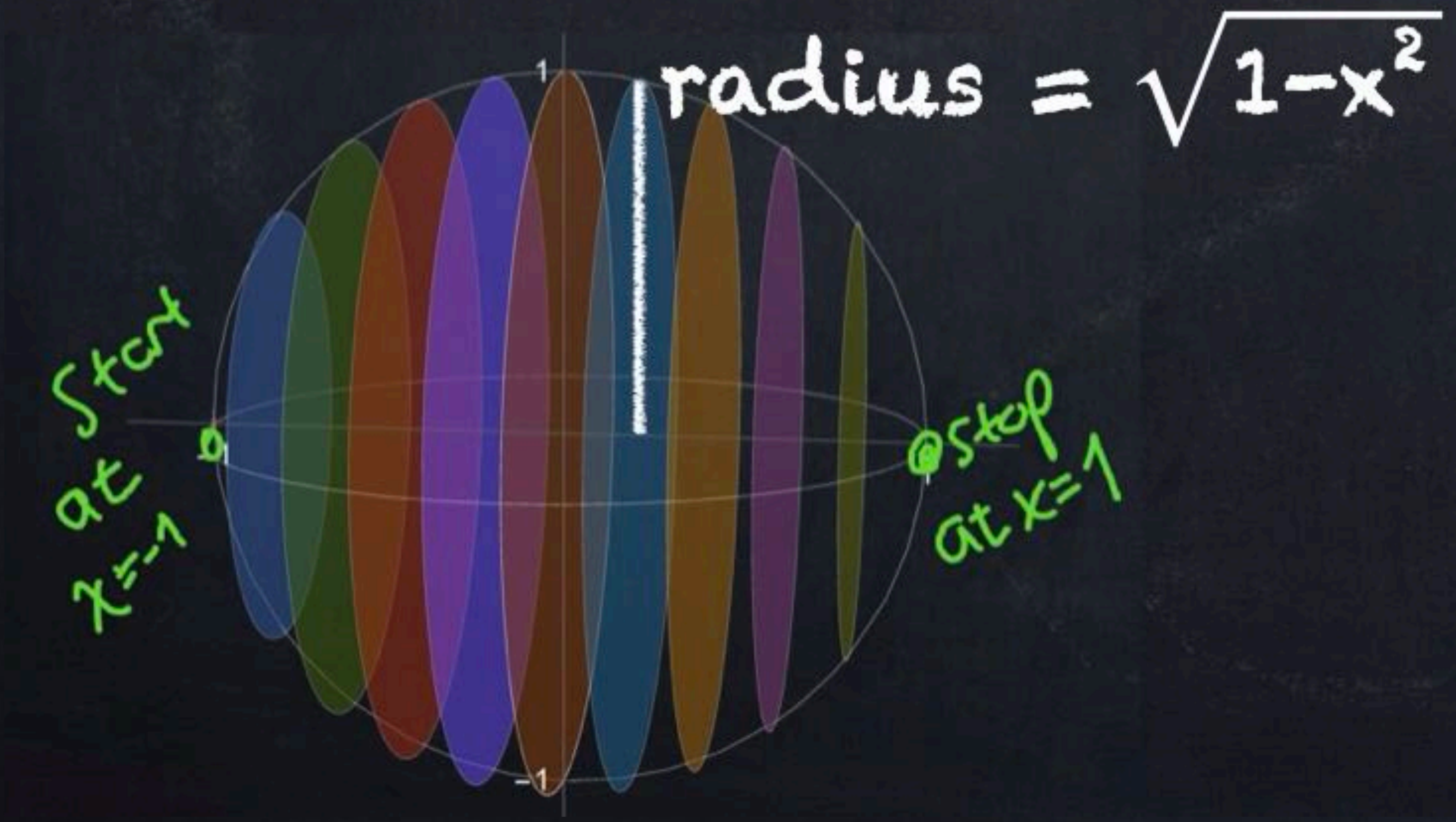
- $$\text{Area} = \int_B^T w(y)\,\mathrm{d}y$$



Likewise, we have two methods for computing a volume of a solid formed by rotating a region around an axis.
I will use the “disk method”, but you can use “cylinders” if you prefer.

Use an integral to find the volume of the sphere formed by rotating $x^2 + y^2 = 1$ around the x-axis.





Answer: $\int_{-1}^{1} \pi\left(\sqrt{1-x^2}\right)^2 dx = \frac{4}{3}\pi$