Analysis 1

24 January 2024

Warm-up: Calculate $\displaystyle\int_1^2 \frac{1}{x^5}\,\mathrm{d}x.$

Integrals

Indefinite integral example: $\int \frac{1}{x^5}\mathrm{d}x = \frac{-1}{4x^4} + C$

Definite integral example: $\int_1^2 \frac{1}{x^5}\mathrm{d}x = \frac{15}{64}$

New vocab: bounds integrand

We have four main techniques to find an indefinite integral:

- just think about derivatives
- use algebra first
- substitution
- parts

Often people think of definite integrals as having an answer that is a number, but this isn't technically correct.

Warm-up: $\displaystyle\int_1^2 \frac{1}{x^5}\mathrm{d}x = \frac{-1}{4(2)^4} - \frac{-1}{4(1)^4} = \frac{15}{64}$

New task: Calculate $\displaystyle\int_1^b \frac{1}{x^5}\mathrm{d}x$.

Last week

Task 1: $\int \underbrace{2x}_{u} \underbrace{\cos(3x)\,\mathrm{d}x}_{dv}$

works the first time

Task 2: $\int \underbrace{4e^{2x}}_{dv} \underbrace{x^2}_{u} \underbrace{\mathrm{d}x}_{dv}$

requires ∫ by parts twice

Today

Task: $I = \int e^{3x}\cos(x)\,\mathrm{d}x.$

use parts twice and then solve equation for I

Task: $\int e^{3x}\cos(x)\,\mathrm{d}x. = \frac{3}{10}e^{3x}\cos(x) + \frac{1}{10}e^{3x}\sin(x) + C$

Task 2: $\int \ln(x)\,\mathrm{d}x. = x\ln(x) - x + C$

Rational functions

The "inverse tangent" function $\arctan(x)$ satisfies

$$\tan(\arctan(x)) = x \qquad \text{and} \qquad \big(\arctan(x)\big)' = \frac{1}{x^2+1}.$$

We can use this for some integrals:

$$\int \frac{16}{x^2+1}\,\mathrm{d}x = 16\arctan(x) + C$$

$$\int \frac{16x}{x^2+1}\,\mathrm{d}x = 8\ln(x^2+1) + C$$

$$\int \frac{1}{16x^2+1}\,\mathrm{d}x = \frac{1}{4}\arctan(4x) + C$$

Rational functions

The integral of any "rational function" $\dfrac{\text{polynomial}}{\text{polynomial}}$ can be done using

- inverse tangent,
- natural logarithm,
- and a lot of algebra (for example, partial fractions).

$$\int \frac{12x^2+x+9}{3x^3+3x^2+2x+2}\mathrm{d}x = \int \frac{12x^2+x+9}{(x+1)(3x^2+2)}\mathrm{d}x = \int \left(\frac{4}{x+1}+\frac{1}{3x^2+2}\right)\mathrm{d}x$$

$$= 4\left(\int \frac{1}{x+1}\mathrm{d}x\right) + \frac{1}{2}\left(\int \frac{1}{\frac{3}{2}x^2+1}\mathrm{d}x\right) = 4\ln(x+1) + \frac{1}{\sqrt{6}}\arctan\left(\sqrt{\frac{3}{2}}\,x\right) + C$$

$w = x+1$ $\quad$ $u = \sqrt{\frac{3}{2}}\,x$

Improper integrals

An **improper integral of the first kind** is a definite integral where one or both of the integral "bounds" are infinite.

Official definitions:

- $$\int_a^{\infty} f(x)\,\mathrm{d}x \quad \text{means} \quad \lim_{b\to\infty}\int_a^b f(x)\,\mathrm{d}x.$$
- $$\int_{-\infty}^{b} f(x)\,\mathrm{d}x \quad \text{means} \quad \lim_{a\to-\infty}\int_a^b f(x)\,\mathrm{d}x.$$
- $$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x \quad \text{means} \quad \lim_{a\to-\infty}\int_a^c f(x)\,\mathrm{d}x + \lim_{b\to\infty}\int_c^b f(x)\,\mathrm{d}x.$$

Improper integrals

An **improper integral of the first kind** is a definite integral where one or both of the integral "bounds" are infinite.

- Example: Calculate $\int_1^\infty \frac{1}{x^5}\mathrm{d}x.$

Officially, this is shorthand for $\lim_{b\to\infty}\int_1^b \frac{1}{x^5}\mathrm{d}x.$

Answer: $\frac{1}{4}$

Task 1: $\int_{\pi}^{\infty} \cos(x)\, \mathrm{d}x$

$\lim_{b\to\infty} \sin(b)$ does not exist.

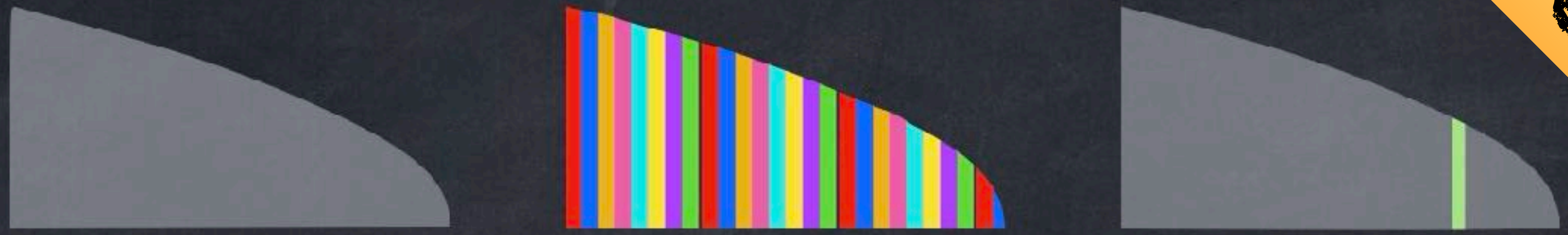
Task 2: $\int_{-\infty}^{0} xe^{x}\, \mathrm{d}x.$

Indefinite using parts: $\int xe^x dx = xe^x - \int e^x dx = e^x(x-1) + C$

For improper, this is $\lim_{a\to-\infty} e^0(0-1) - e^a(a-1) = -1$

requires L'Hospital's Rule

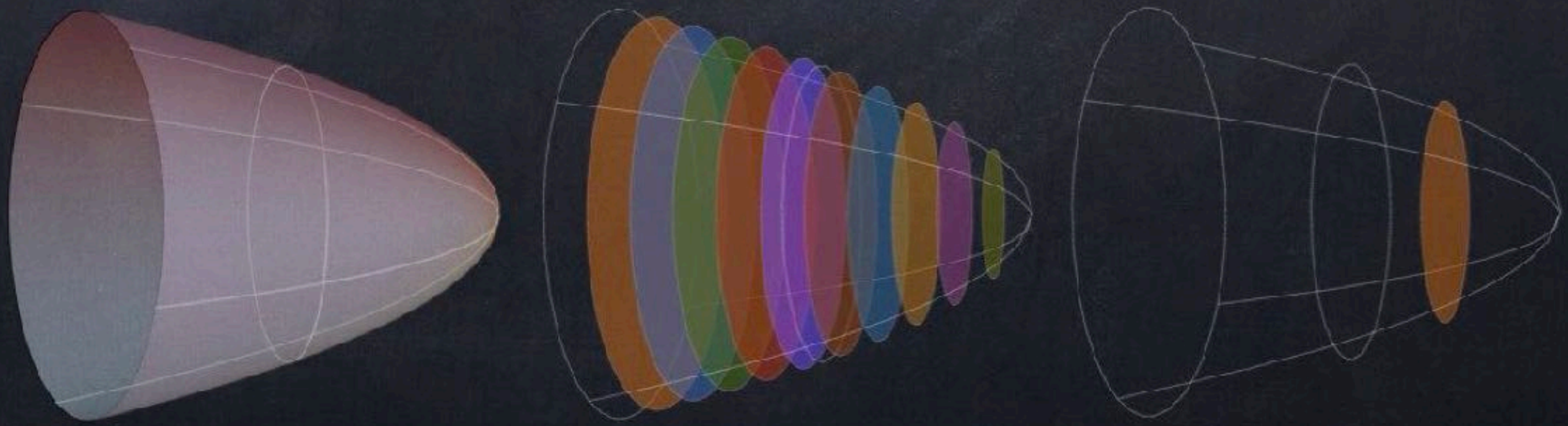
$$A = \int_a^b h(x)\,\mathrm{d}x$$



Volume as an integral

$$V = \int_a^b A(x)\,\mathrm{d}x$$

$$V = \int_a^b \pi\big(r(x)\big)^2\,\mathrm{d}x$$



Find the volume of the solid formed by rotating the region with $2 \leq x \leq 7$ and $0 \leq y \leq \frac{1}{2}x$ around the x-axis.

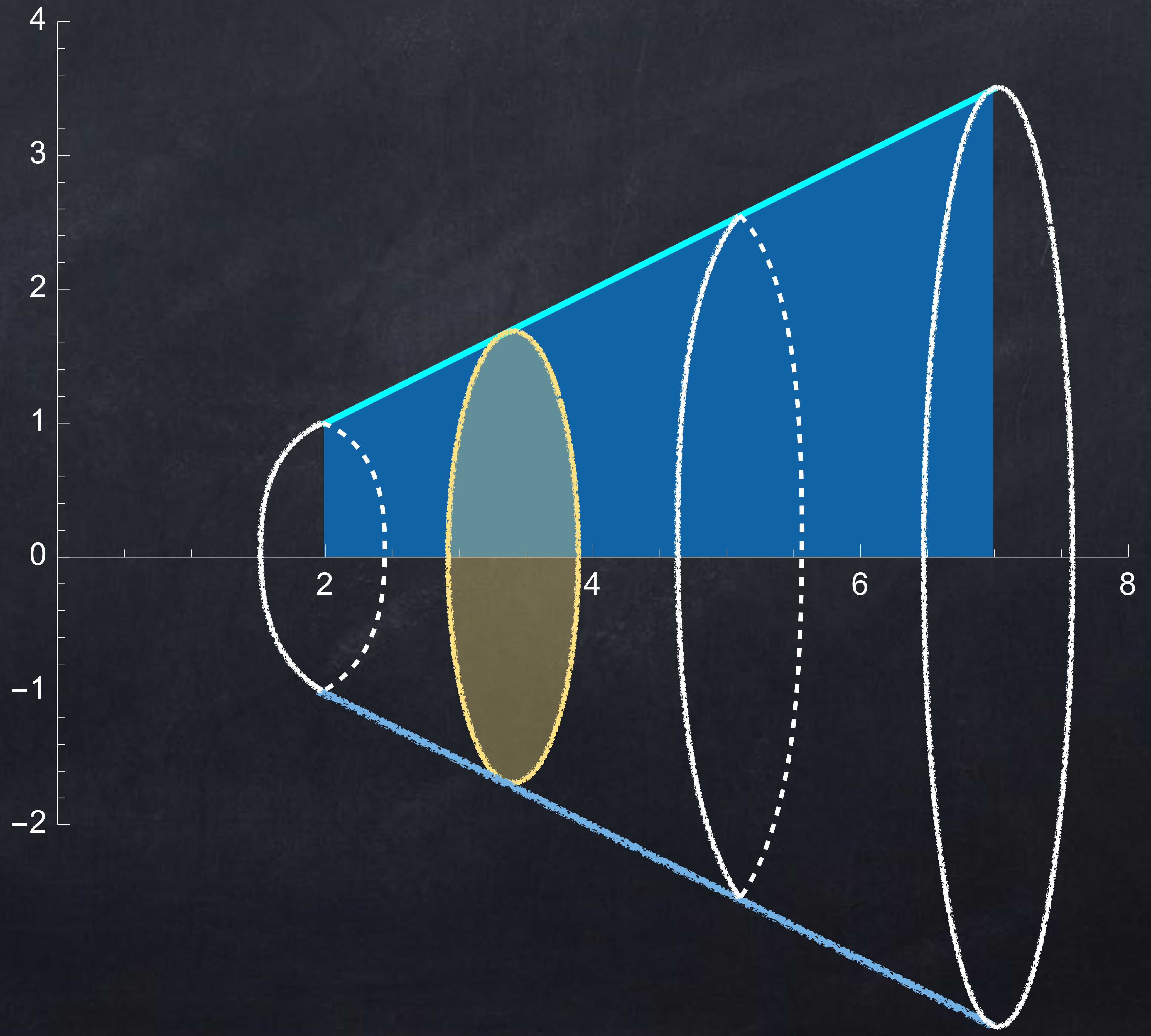
Last week

$$\text{Volume} = \int_{\text{left } x}^{\text{right } x} \text{Area } dx$$

$$= \int_2^7 \pi (\text{radius})^2 \, dx$$

$$= \int_2^7 \pi \left(\frac{x}{2}\right)^2 dx$$

$$= \frac{(7)^3}{12}\pi - \frac{(2)^3}{12}\pi = \frac{355}{12}\pi$$



Just like with area, some volumes are easier to calculate using $\int \ldots \mathrm{d}y$.

Task: Rotate the region bound by $y = 11 - x^2$ and $y = 5$ around the y-axis. Find the volume of this solid.

$$\int_5^{11} \pi r^2 dy = \ldots = \boxed{18\pi}$$

Volumes with holes/gaps in them can often be found using the “washer method” (named after the hardware piece—Polish *podkładka*—not the machine for cleaning clothes).

You will not need to use this on any quiz or exam in this course.

Celebration of Knowledge #2

The final exam will be

Wed. 7 February at 12:00 noon

room 201 / C-1

Second attempt one week later.

What topics would be most helpful to review next week?

- Limits (including L'Hospital's Rule)
- Derivative rules
- Tangent lines
- CP, min, max
- Inflection points
- Taylor polynomials
- Basic integrals
- $\int$ by substitution
- $\int$ by parts
- Area
- Volume