Analysis 1

22 November 2023

Warm-up: For what value(s) of x does

$$\frac{1}{2}x^3 = 4 \, ?$$

Celebration of Knowledge 🥳

(Midterm exam)

Topics:

- limits of sequences
- limits of functions
- discontinuities / recognizing graphs
- derivatives (Power, Sum, CM rules only)
- tangent lines

See List 4.

Last time

Idea: $f'(5)$ is a *number* that is the slope of the tangent line to $y = f(x)$ through the point $(5, f(5))$.

Idea: $f'(x)$ is a *function* that gives the derivative for various x-values.

- Also written f' $\quad Df$ $\quad \dfrac{\mathrm{d}}{\mathrm{d}x}f$ $\quad \dfrac{\mathrm{d}f}{\mathrm{d}x}$ $\quad \dfrac{\mathrm{d}y}{\mathrm{d}x}$

Calculations: We can find formulas using "derivative rules":

- For any constant c and function f, $\quad (cf)' = c \cdot f'$.
- For any constant p and function f, $\quad (x^p)' = px^{p-1}$.
- For any functions f and g, $\quad (f+g)' = f' + g'$

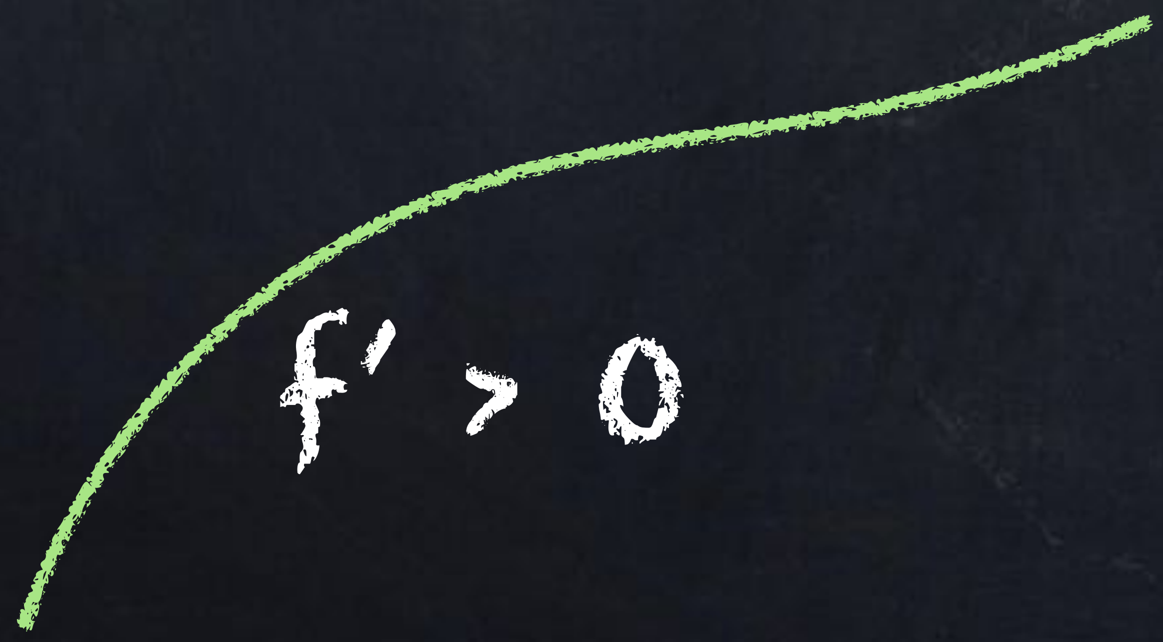
Increasing and decreasing

Last time

Definitions: We say $f(x)$ is strictly **increasing** on an interval if for any a, b in that interval with $a < b$ we have $f(b) > f(a)$.

We say $f(x)$ is strictly **decreasing** on an interval if … $f(b) < f(a)$.

Facts: If $f(x)$ is strictly increasing on an interval, then $f'(x) > 0$ for all x-values in that interval. If $f(x)$ is strictly decreasing, then $f'(x) < 0$.



Critical points

A **critical point** of f is an x-value where $f'(x)$ is either zero or doesn't exist.

- zero → horizontal tangent line
- does't exist → vertical tangent line, or corner, or discontinuity

A function can only change from increasing to decreasing (or dec. to inc.) at a critical point.

Example 1: Find the critical point(s) of $f(x) = x + 9x^{-1} - 5$.

Answer: $x = -3$, $x = 0$, $x = 3$

Example 2: Find the critical point(s) of $f(x) = \frac{1}{8}x^4 - 4x + 3$.

Warmup: $\frac{1}{2}x^3 - 4 = 0$

Answer: $x = 2$

Example 2: Find the critical point(s) of $f(x) = \frac{1}{8}x^4 - 4x + 3$.

$x = 2$ only

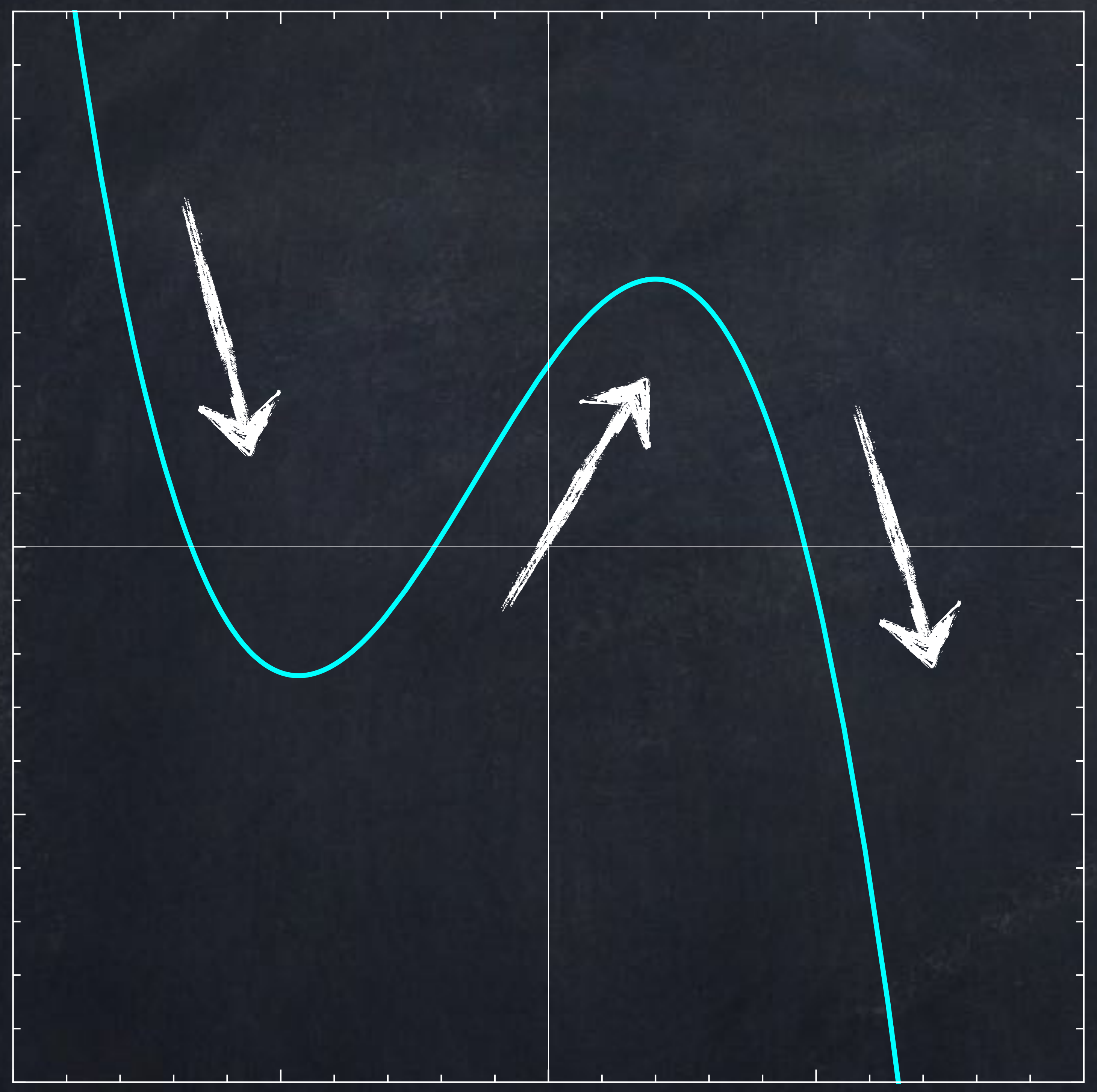
- On what interval(s) is $f(x) = \frac{1}{8}x^4 - 4x + 3$ increasing?

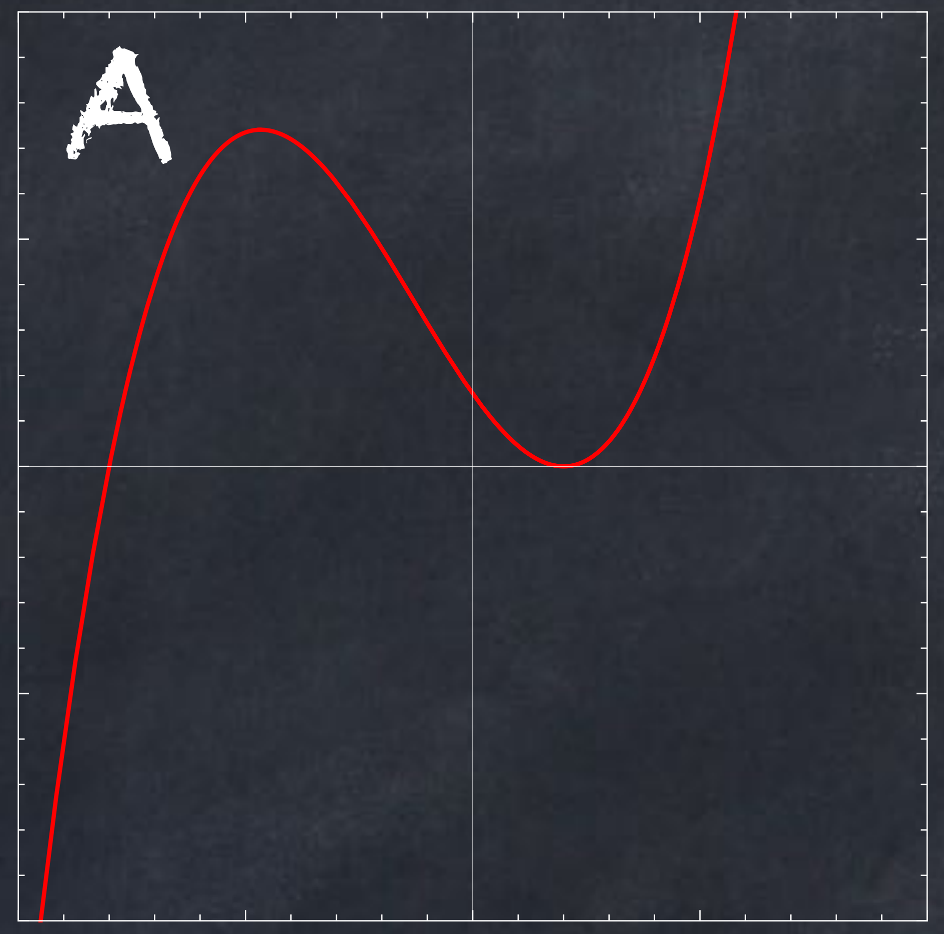
$x > 2$ or $(2, \infty)$

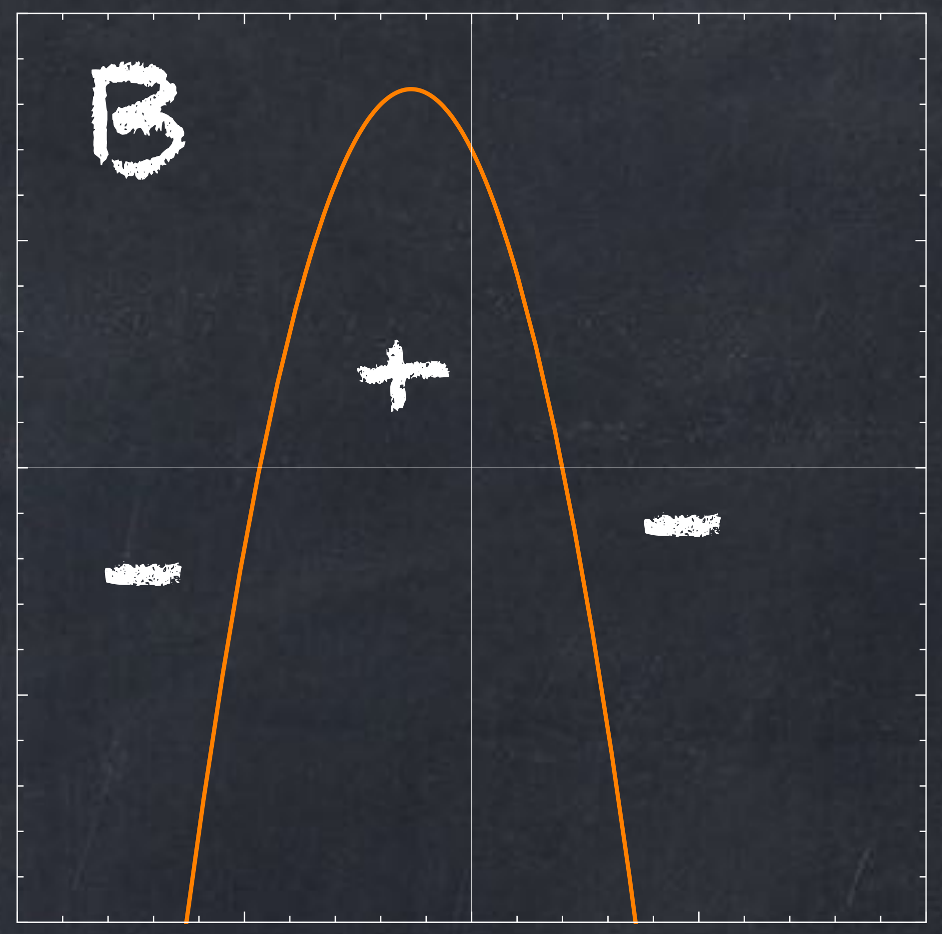
- On what interval(s) is $f(x) = \frac{1}{8}x^4 - 4x + 3$ decreasing?

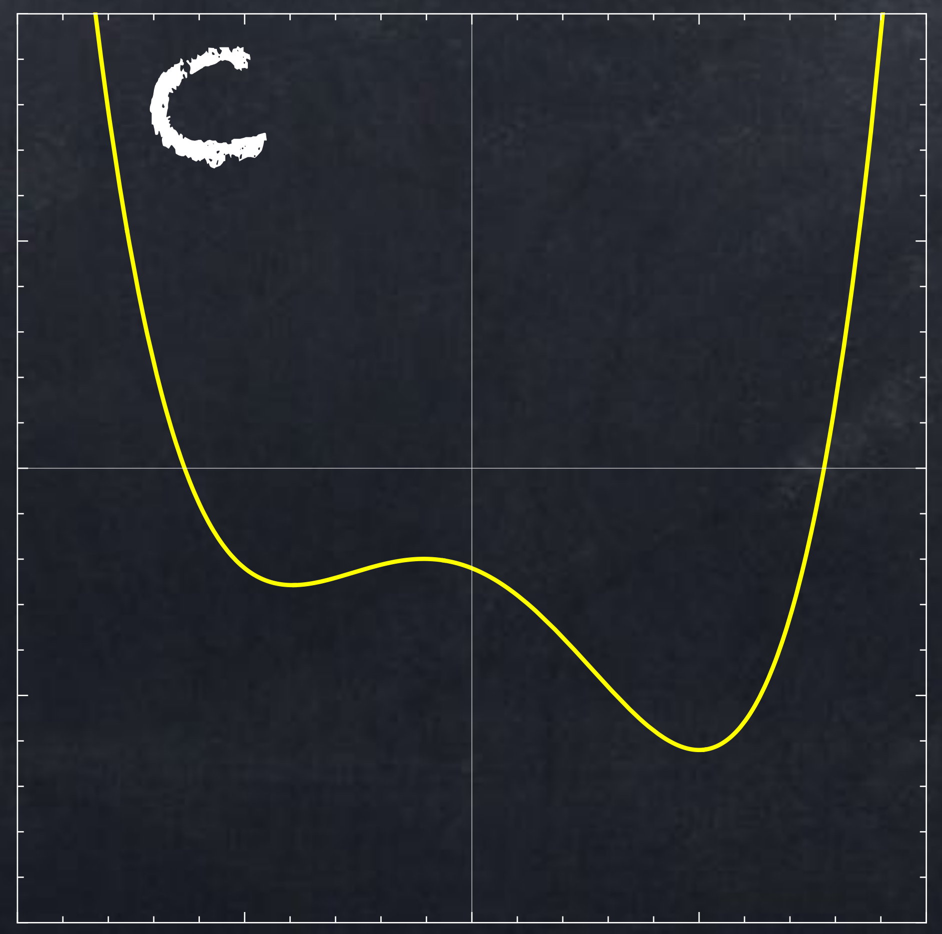
$x < 2$ or $(-\infty, 2)$

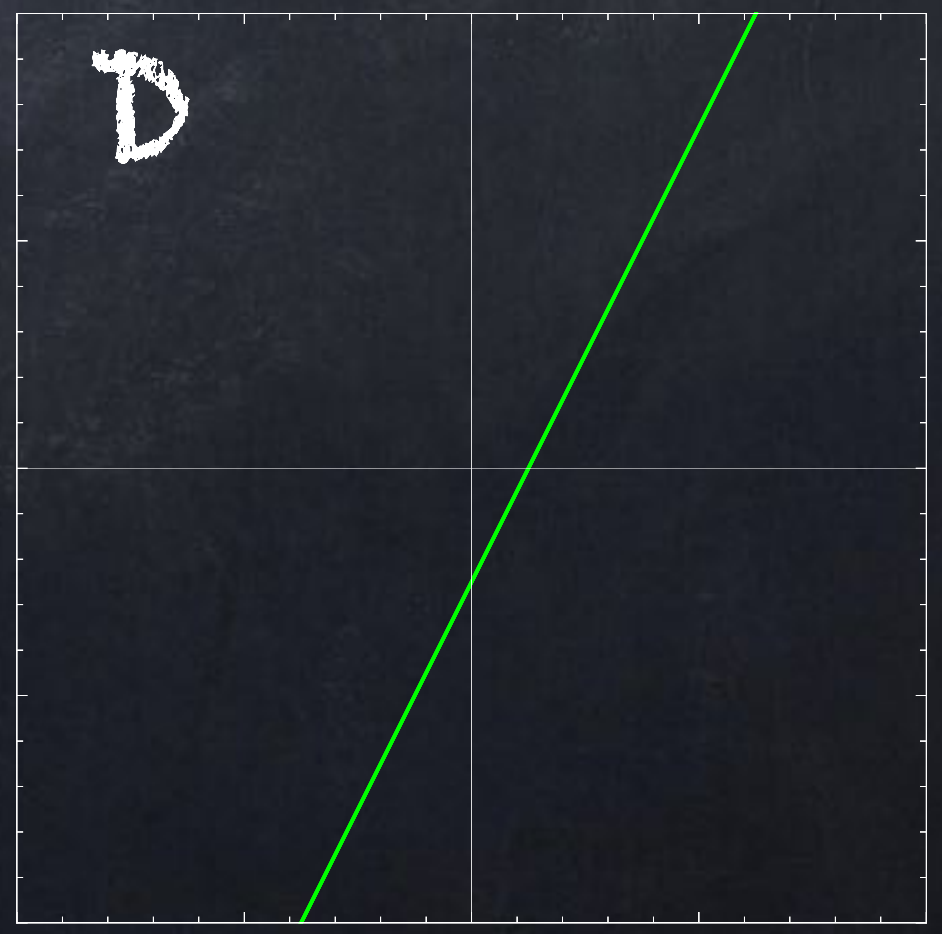
Which of the graphs A-D is the derivative of the graph on the left?



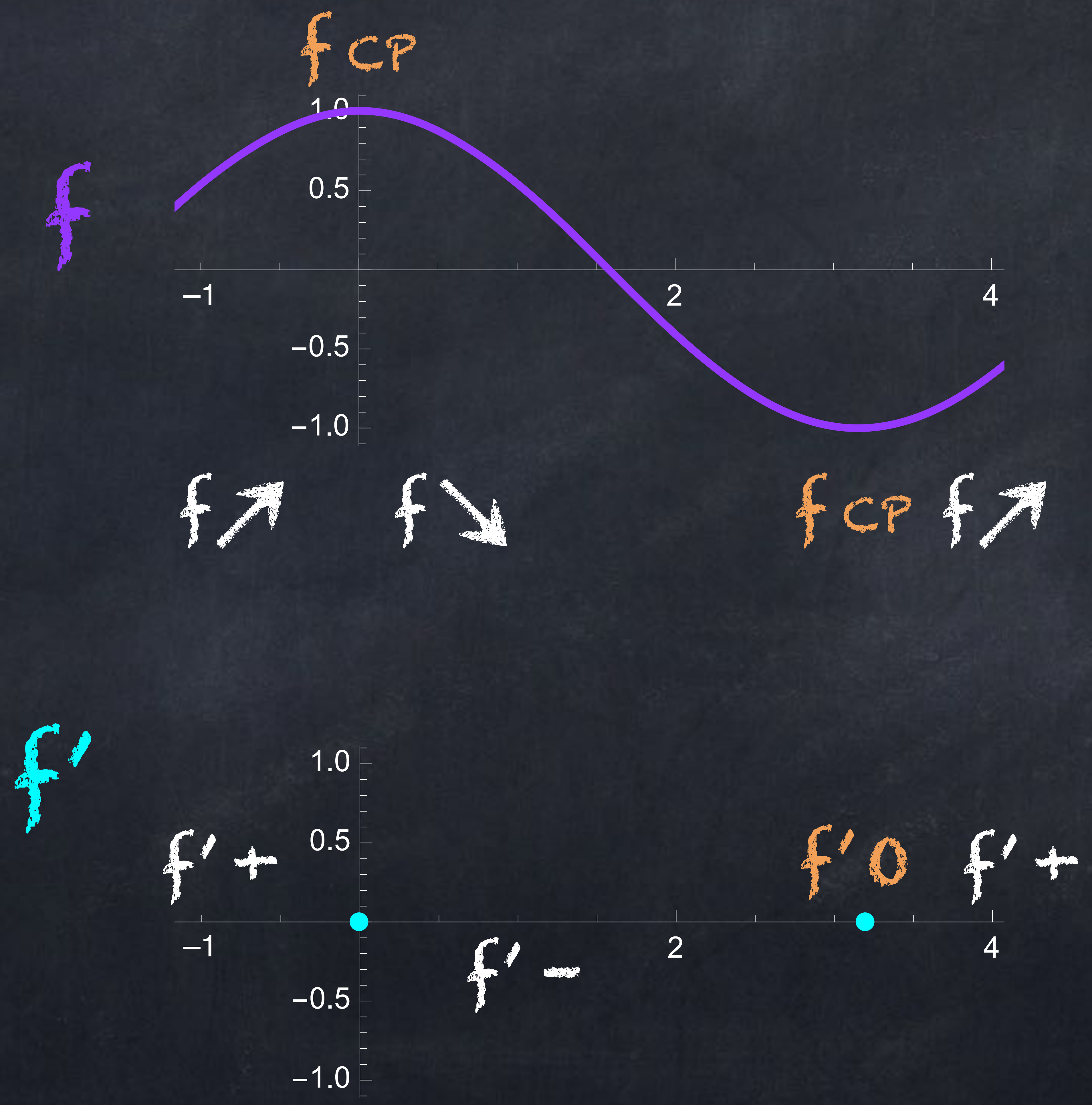








If we have a graph of $y = f(x)$, we can get a good idea of f'.



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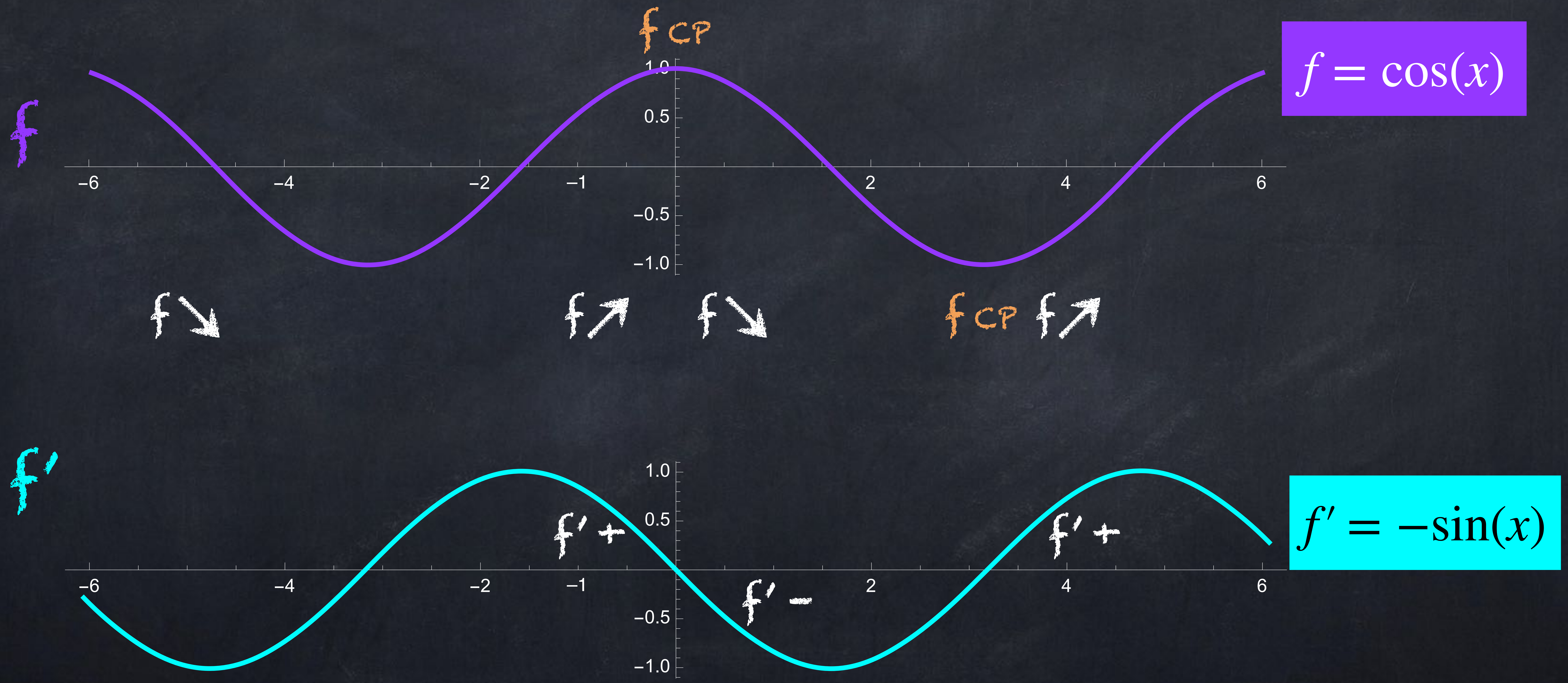


$$\frac{d}{dx}\left[\sin(x)\right] = \cos(x) \qquad \frac{d}{dx}\left[\cos(x)\right] = -\sin(x)$$

If you just remember that the derivative of $\cos(x)$ is some other trig function, the increasing/decreasing intervals can tell you that it must be $-\sin(x)$.

We still have the Constant Multiple Rule and the Sum Rule, so we can also do

- $\frac{d}{dx}\left[x^2 + 190 + 2\sin(x)\right] = 2x + 2\cos(x)$,
- $\left(4x^3 + 6\cos(x) + x\right)' = 12x^2 - 6\sin(x) + 1$.

We do not *yet* have a rule to find $\frac{d}{dx}\left[\sin(2x)\right]$ or $\left(\tan(x)\right)'$.

Critical points

A number c is a "critical point" of $f(x)$ …

- if $f'(c) = 0$ (horizontal tangent line) or
- if $f'(c)$ doesn't exist (vertical tangent line, or corner, or discontinuity).

Increasing and decreasing

On an interval *or* at a single point,

- if $f' > 0$ then f is increasing,
- if $f' < 0$ then f is decreasing.

Minimum and maximum

- How do these relate to derivatives?

Extrema (min and max)

We say $f(x)$ has…

- an **absolute maximum at $x = c$** if $f(c) \geq f(x)$ for all allowed x values.
- an **absolute minimum at $x = c$** if $f(c) \leq f(x)$ for all allowed x values.
- a **local maximum at $x = c$** if $f(c) \geq f(x)$ for all x in some open interval containing c.
- a **local minimum at $x = c$** if $f(c) \leq f(x)$ for all x in some open interval containing c.

- an **absolute extreme** if it has a absolute max *or* absolute min.
- a **local extreme** if it has a local max *or* local min.

Extrema (min and max)

The plural of maximum is *maxima* or *maximums*.
Or you can just write “maxs” or “max”.

1 minimum (or 1 min) → 2 minima or 2 minimums or 2 mins or 2 min.

1 extremum (or 1 extreme) → 2 extrema or 2 extremes.

Types of extremes

- Local maximum(s)

  A C E

- Local minimum(s)

  B D

- Absolute maximum(s)

  C E

- Absolute minimum(s)

  (none)

Types of extremes

On the interval
$0 \leq x \leq 3$ only.

- Absolute maximum(s)

  c (x = 1)

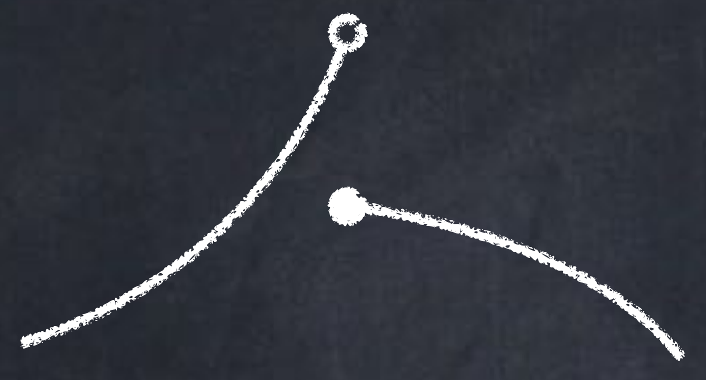
- Absolute minimum(s)

  x = 3, y = −1

Extreme Value Theorem

If $f(x)$ is continuous on $[a, b]$ then it has at least one absolute min and at least one absolute max somewhere in $[a, b]$.

Continuous is necessary:



If $f(x)$ has a local extreme at $x = c$ then c is a critical point of f.

The opposite is not true: a fn. might have critical points that are not extremes (example: x^3 at $x=0$).

Finding absolute extremes

To find the absolute extremes of a continuous function $f(x)$ on a closed interval $[a, b]$,

1. Find the critical points. *Ignore any that don't satisfy* $a \leq x \leq b$*.*

2. Compute the value of the function at the points from Step 1.

3. Compute the value of the function at the endpoints (a and b).

4. The largest f-value from Steps 2 and 3 is where you have abs. max.
   The smallest f-value from Steps 2 and 3 is where you have abs. min.

Absolute min and max

Find the absolute extrema of $f(x) = x - \sqrt{x}$ on the closed interval $0 \leq x \leq 2$.

$$f' = 1 - \frac{1}{2\sqrt{x}}$$

CP are $x = 0$ and $x = 1/4$

x	f	
0	0	← absolute min
1/4	2	← absolute max
2	$2-\sqrt{2} \approx 0.59$	

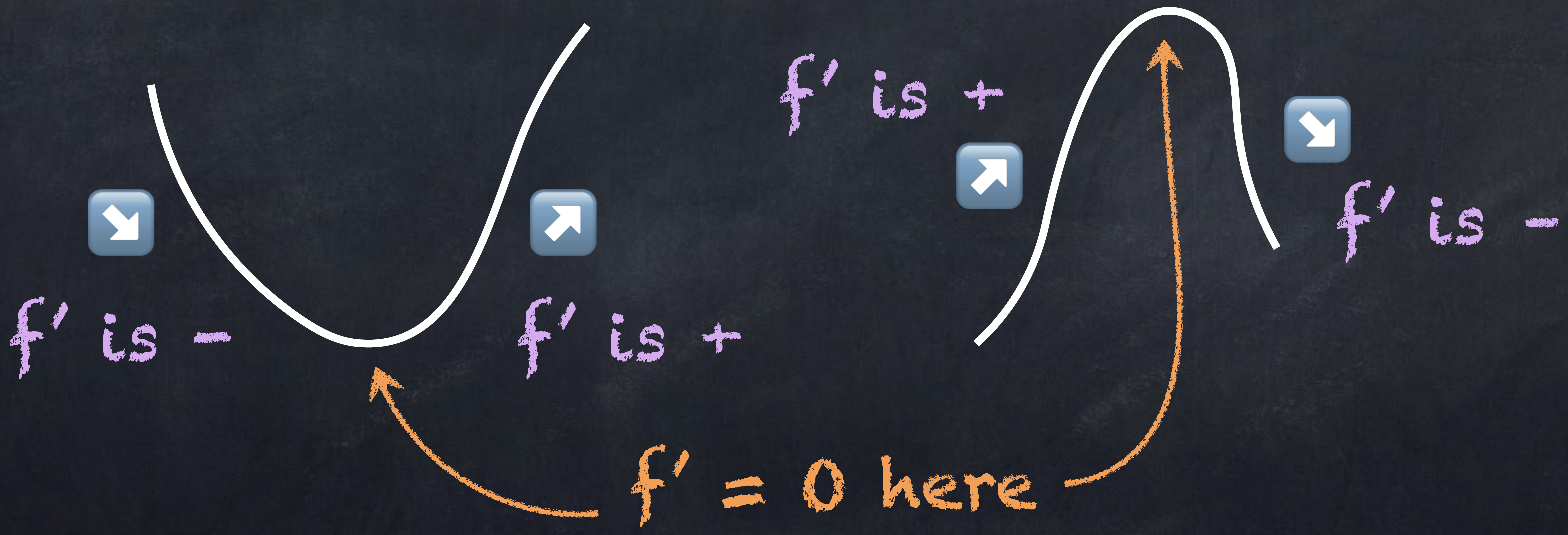
Find the absolute extremes of $g(x) = \sqrt{3}\sin(x) + 3\cos(x)$ with $0 \leq x \leq \pi$.

Final answer: abs. min at $x = \pi$ and abs. max at $x = \frac{\pi}{6}$.

Finding local extremes

Now we know how to find absolute extremes (on an interval).

What about local extremes?



Finding local extremes

To find the local min/max of $f(x)$,

1. Find the critical points of f.

2. Compute signs of f' somewhere in between each CP, and at one point with $x <$ all CP, and at one point with $x >$ all CP.

3. **The First Derivative Test**

- If $f' > 0$ to the left of $x = c$ and $f' < 0$ to the right of $x = c$, then f has a local maximum at $x = c$.
- If $f' < 0$ to the left of $x = c$ and $f' > 0$ to the right of $x = c$, then f has a local minimum at $x = c$.
- If f' has the same sign on both sides of $x = c$, then f has *neither* a local minimum nor local maximum at $x = c$.

Example 1: Given that the critical points of

$$g(x) = \frac{1}{5}x^5 - 2x^3 + 4x^2 - 3x$$

are -3 and 1, classify each as a local minimum, local maximum, or neither.

$$g' = x^4 - 6x^2 + 8x - 3$$

x		-3		1	
g	↗	local max	↘	local min	↗
g'	$+$	0	$-$	0	$+$

Example 2: Given that the critical points of

$$f(x) = \frac{x^5}{5} + x^4 - \frac{2x^3}{3} - 6x^2 + 9x$$

are -3 and 1, classify each as a local minimum, local maximum, or neither.

$$f' = x^4 + 4x^3 - 2x^2 - 12$$

x		-3		1	
f	↗	neither	↗	neither	↗
f'	25	0	9	0	25