Analysis 1

13 December 2023

Warm-up: Activity in hallway
with $\cdots \to f \to f' \to \cdots$ papers.

Yes-terday

L'Hospital's Rule

- If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$ then $\lim_{x\to a} \frac{f}{g} = \lim_{x\to a} \frac{f'}{g'}$.
- There are similar statements for $\pm\frac{\infty}{\infty}$, and for $\lim_{x\to a^{\pm}}$, and for $\lim_{x\to\pm\infty}$.

Taylor polynomials / Taylor series

- $P(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$ guarantees that $P(a) = f(a)$ and $P'(a) = f'(a)$ and … up to $P^{(N)}(a) = f^{(N)}(a)$.
- Series: $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

Task 1: Find the degree 3 Taylor polynomial of

$$f(x) = \cos(\ln(x))$$

around $x = 1$.

$f(x) = \cos(\ln(x))$ ← also $f^{(0)}$

$f'(x) = \dfrac{-\sin(\ln(x))}{x}$ ← also $f^{(1)}$

$f''(x) = \dfrac{-\cos(\ln(x)) + \sin(\ln(x))}{x^2}$

$f^{(3)}(x) = \dfrac{3\cos(\ln(x)) - \sin(\ln(x))}{x^3}$

$$P(x) = \sum_{k=0}^{3} \frac{f^{(k)}(1)}{k!}(x-1)^k$$

$$= \frac{f(1)}{0!} + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3$$

$$= \frac{1}{1} + \frac{0}{1}(x-1) + \frac{-1}{2}(x-1)^2 + \frac{3}{6}(x-1)^3$$

$$= \boxed{1 - \frac{1}{2}(x-1)^2 + \frac{1}{2}(x-1)^3}$$

Note: $f^{(0)}$ means f, and $0!$ is 1.

Task 1: Find the degree 3 Taylor polynomial of

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around $x = 1$.

$$P(x) = 1 - \frac{1}{2}(x-1)^2 + \frac{1}{2}(x-1)^3$$

Task 2: Use this polynomial to approximate the number $\cos(\ln(1.2))$.

$$P(1.2) = 1 - \frac{1}{2}(1.2-1)^2 + \frac{1}{2}(1.2-1)^3$$

$$= 1 - \frac{0.04}{2} + \frac{0.008}{2} = \boxed{0.948}$$

The exact value $f(1.2) = 0.983425...$, so 0.948 is a good approximation. Some calculators actually use Taylor polynomials (with very high degree) when asked for "cos" and "ln" values.

(This whole slide is <u>not</u> required for this class. It's just a neat application.)

Taylor series also appear as "generating functions" when studying sequences.

Example: Fibonacci $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$ has $a_n = a_{n-1} + a_{n-2}$.

By working with

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = x + x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 + 13x^7 + \cdots$$

it is possible to show $f(x) = \dfrac{-x}{x^2 + x - 1}$ and then use that to prove

$$a_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

Note $\dfrac{1 \pm \sqrt{5}}{2}$ are the roots of $x^2 + x - 1$.

Anti-derivative

An **anti-derivative** of f is a function whose derivative is f.

So "$F(x)$ is an anti-derivative of $f(x)$" means that $F'(x) = f(x)$.

Example: Each of the following is an anti-derivative of $2x$:

- x^2
- $x^2 + 1$
- $x^2 - 28$
- $x^2 + \sqrt{51}$
- $x^2 - 0.387$

Describe *all* anti-derivatives of $10x^4$.

- That is, describe *all* functions $F(x)$ that have $F'(x) = 10x^4$.

$2x^5$ + any constant

It's common to write this as $2x^5 + C$.

Notation (how to write math)

	Newton	Leibniz	Euler / Lagrange
Derivative of f	$\dot{f}$	$\frac{\mathrm{d}f}{\mathrm{d}x}$	f' or $f^{(1)}$
Anti-derivative of f	$\acute{f}$ or $\boxed{f}$	$\int f \,\mathrm{d}x$	$f^{(-1)}$

All the ways of writing derivatives are still common today.

Only $\int f \,\mathrm{d}x$ is common for anti-derivatives. We will talk more about this notation later.

Give a formula for an anti-derivative of

- $10x^4$ ⟶ $2x^5$
- x^{22}
- x^{-15}
- x^{-1}
- e^x

☆ $\ln(x)$ Answer: x ln(x) - x. But this is too hard for now.

- $\sin(x)$
- $\cos(x)$

☆ $\cos(x^2)$ Literally impossible (for anyone, forever).

Back of paper from warm-up:

Name / ID

Pick ONE of these and write an anti-derivative.

Area

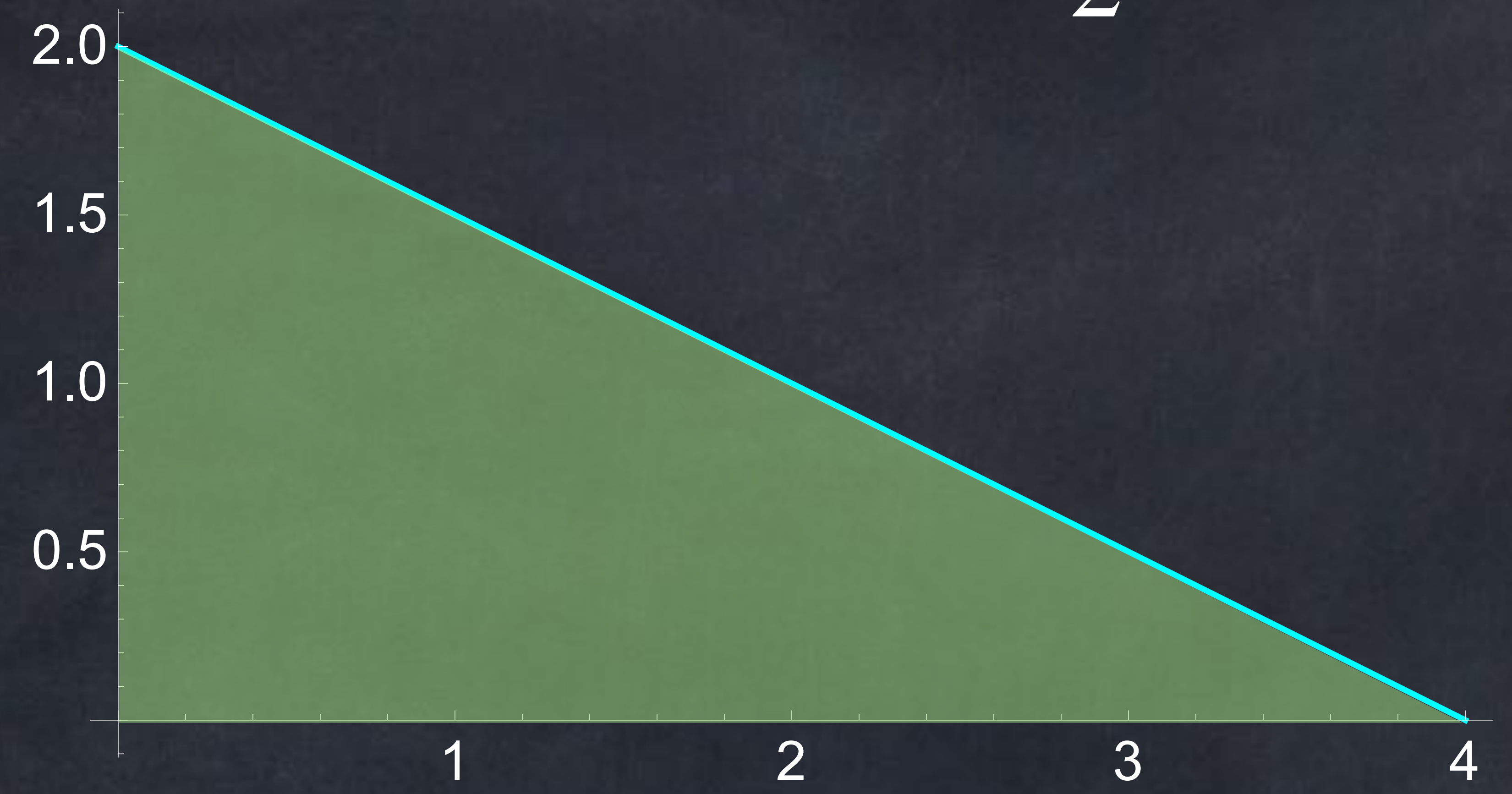
The area of a rectangle is length times width.

What about other shapes?

It is often important to calculate the area “under $y = f(x)$”.
What does this mean?

- Often this looks like  or like  $(a \leq x \leq b)$.
- However, the standard meaning of “area under $y = f(x)$” is that when $f(x) < 0$ we count this as “negative area”.

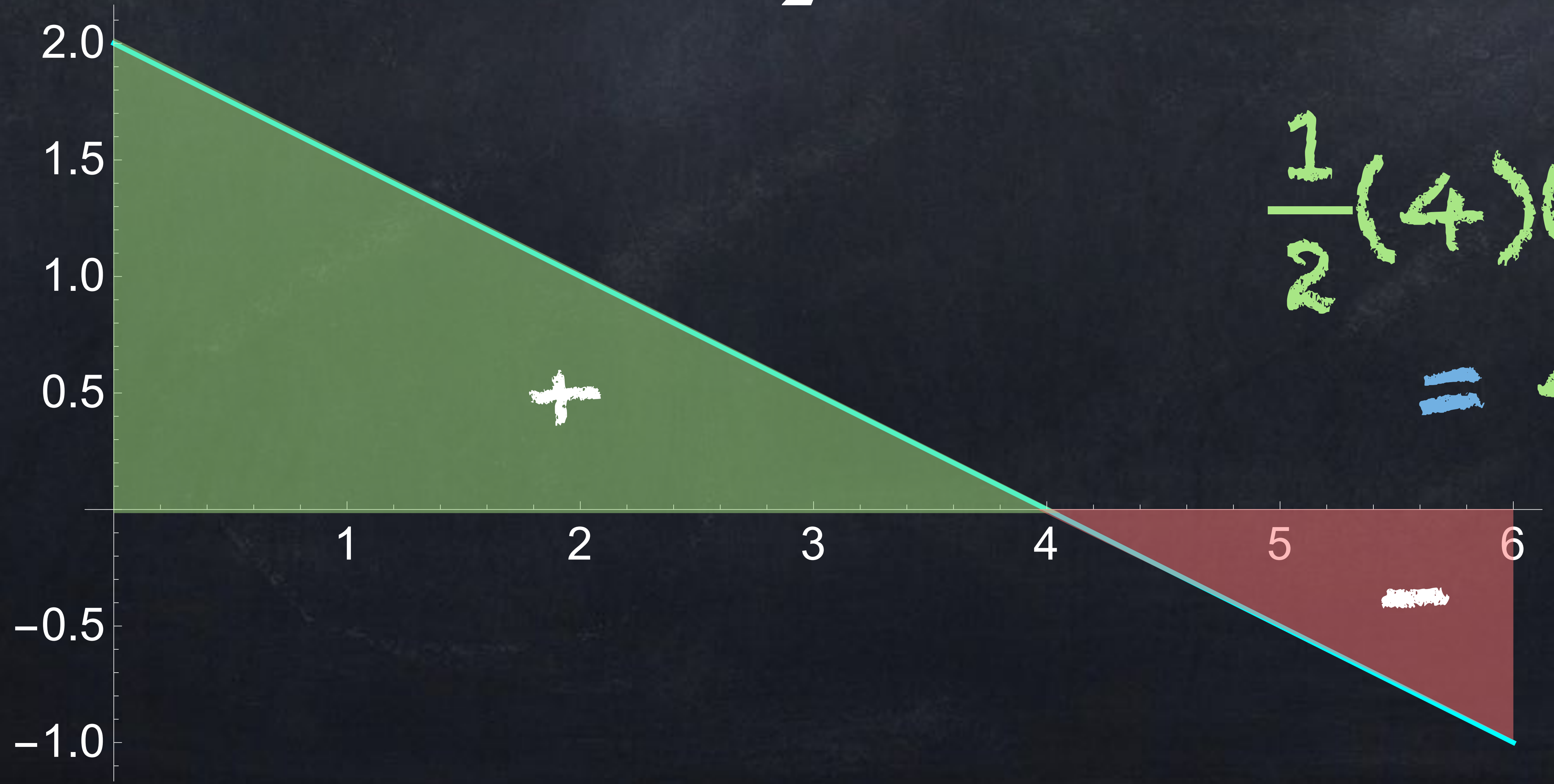
Example: The "area under $y = 2 - \frac{1}{2}x$ from $x = 0$ to $x = 4$" is



$$\frac{1}{2}(\text{height})(\text{base})$$

$$= \frac{1}{2}(4)(2) = 4$$

Example: The "area under $y = 2 - \frac{1}{2}x$ from $x = 0$ to $x = 6$" is



$$\frac{1}{2}(4)(2) + \frac{1}{2}(-1)(2)$$

$$= 4 - 1 = 3$$

Definite Integrals

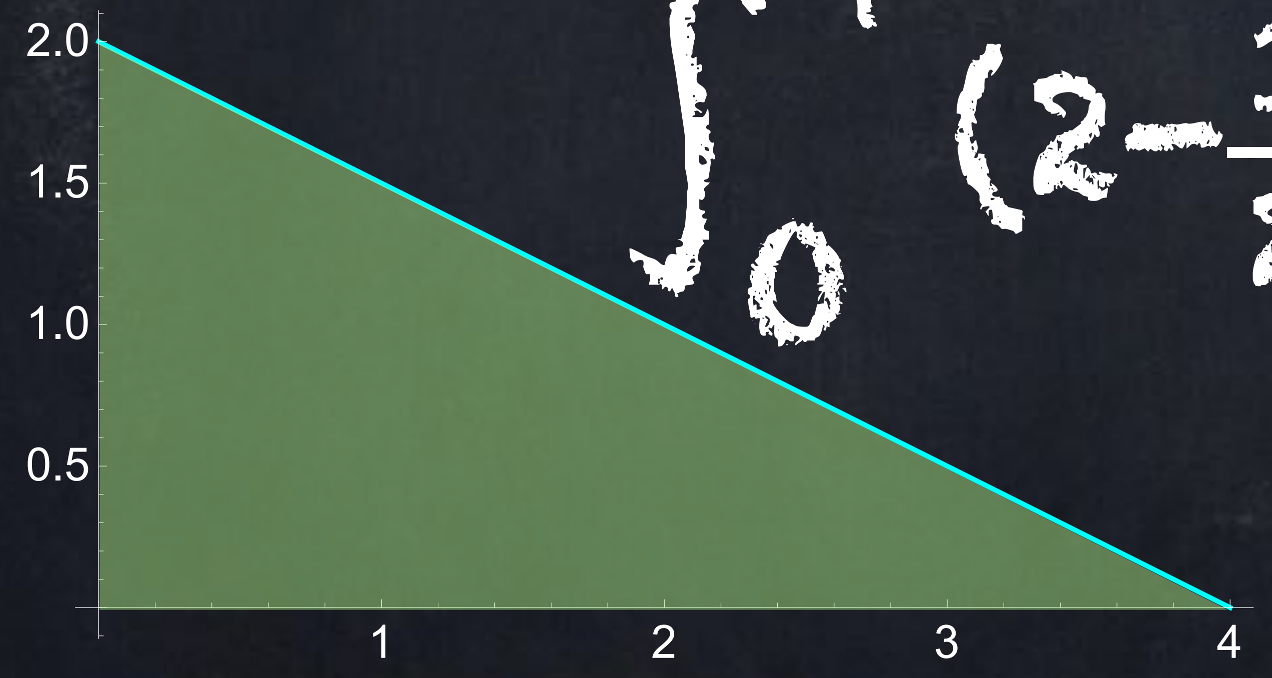
We write

$$\int_a^b f(x)\,\mathrm{d}x$$

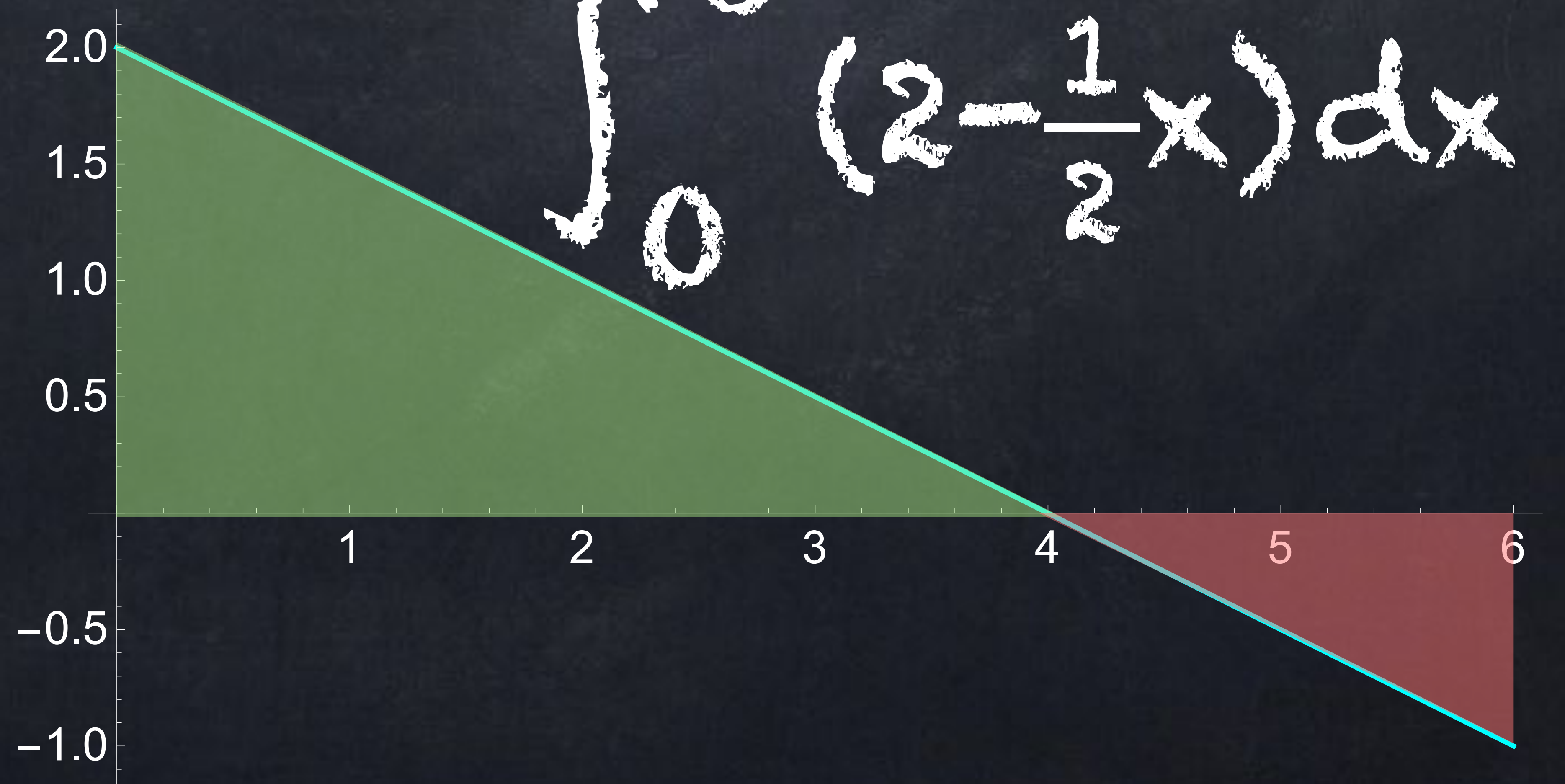
"the integral of f from a to b"

for the area under $y = f(x)$ between $x = a$ and $x = b$.

$$\int_0^4 \left(2-\tfrac{1}{2}x\right)dx = 4$$



$$\int_0^6 \left(2-\tfrac{1}{2}x\right)dx = 3$$



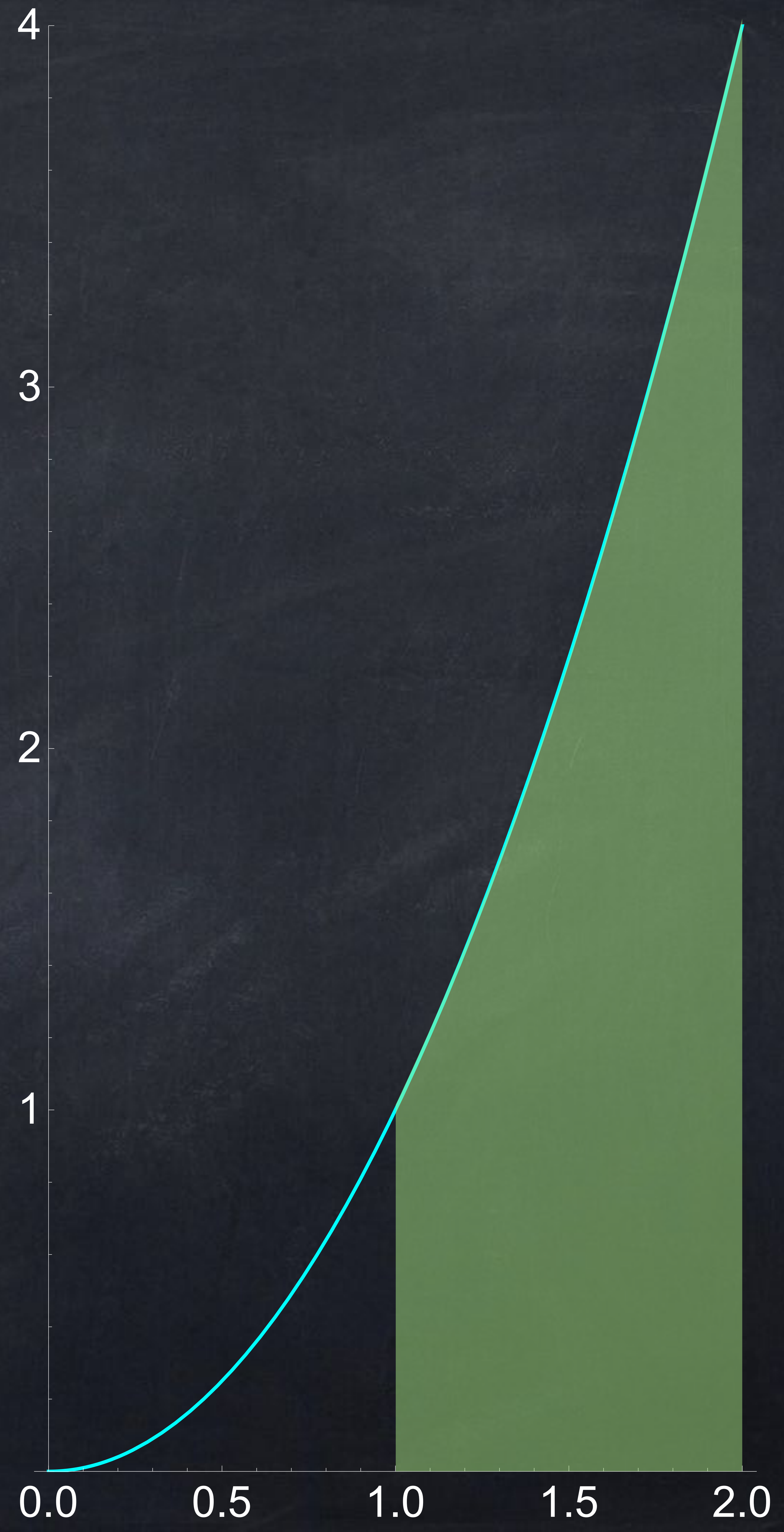
The area under $y = 3$ or $y = x$ or $y = 2 - \frac{1}{2}x$ can be calculated using rectangles and triangles.

The area under $y = x^2$ from $x = 1$ to $x = 2$ is

$$\int_1^2 x^2 \, \mathrm{d}x,$$

but what is this number?

For more complicated functions, we can approximate the area under $y = f(x)$ with rectangles.



The area $\int_1^2 x^2 \mathrm{d}x$ is *approximately* $\sum_{k=1}^{10} \frac{1}{10}\left(1+\frac{k}{10}\right)^2$ and

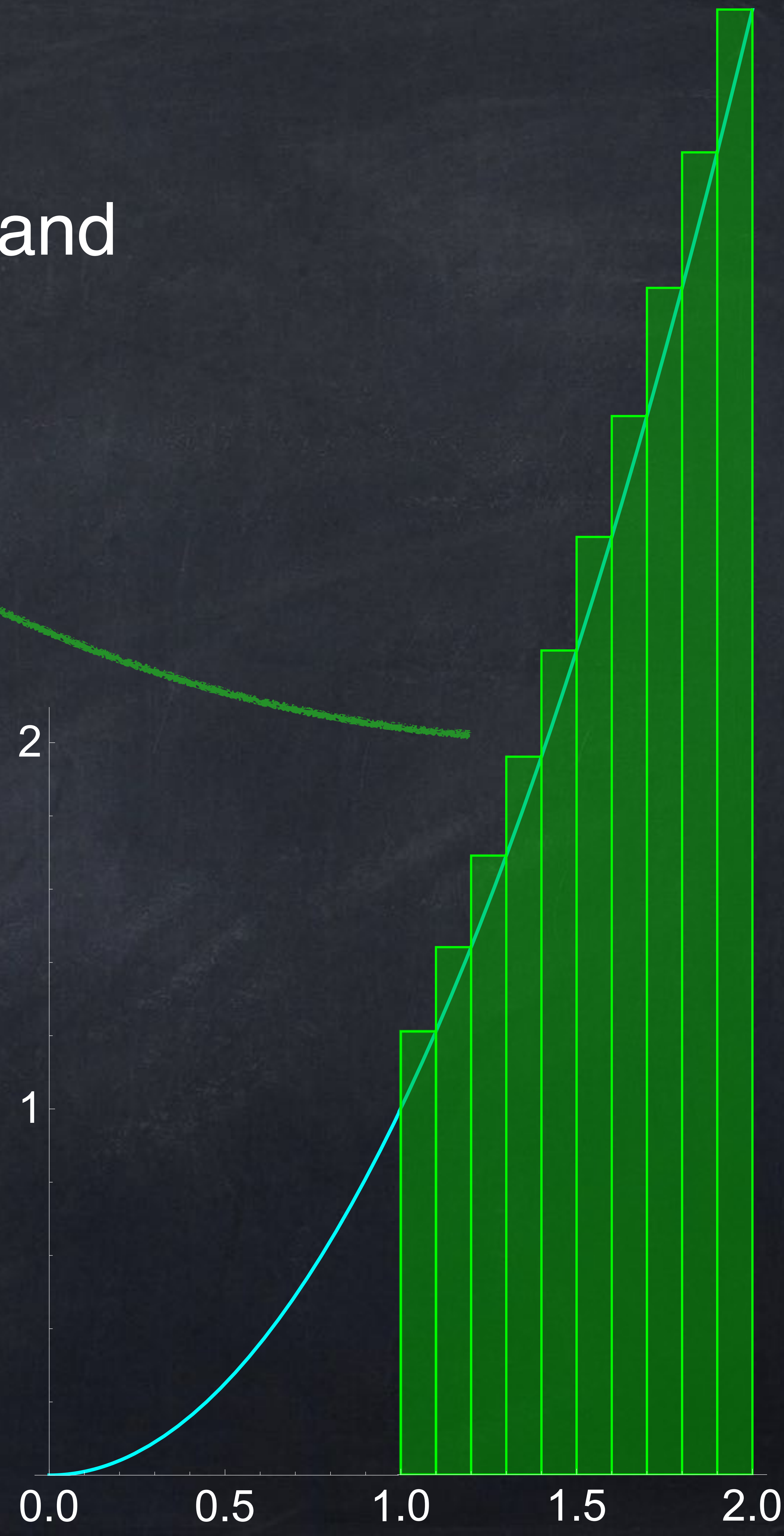
is *exactly* $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n}\left(1+\frac{k}{n}\right)^2$.

- With some work is possible to show that

$$\sum_{k=1}^{n} \frac{1}{n}\left(1+\frac{k}{n}\right)^2 = \frac{14n^2+9n+1}{6n^2}$$

so the area is $\frac{14}{6} = \frac{7}{3}$.

- But there is a much easier way!



Area from anti-derivatives

Instead of using a limit of a sum, there is a very nice way to compute area under a curve using an anti-derivative:

The Fundamental Theorem of Calculus

If f is continuous, then $\displaystyle\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$,

where $F(x)$ is any function for which $F'(x) = f(x)$.

Name means "Important Fact about Analysis"

Because integrals "undo" derivatives, they appear in many places in science and engineering.

voltage $= \int E\, dx$

work $= \int F\, dx$

impulse $= \int F\, dt$

To calculate $\int_1^2 x^2\,\mathrm{d}x$ exactly, we need a function whose derivative is x^2.

- The fact that $\frac{\mathrm{d}}{\mathrm{d}x}(x^2) = 2x$ does not matter. We need the opposite idea.

$F(x) = \frac{1}{3}x^3$ satisfies $F' = x^2$

$F(2) - F(1) = \frac{1}{3}(2)^3 - \frac{1}{3}(1)^3 = \boxed{\frac{7}{3}}$

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$$
with $F' = f$