Analysis 1, Summer 2023

List 2

Limits of functions

35. Use the facts
$$0 < \ln(n) \qquad \text{for all } n \in \mathbb{N} \text{ with } n \geq 2$$
and
$$\ln(n) < \sqrt{n} \qquad \text{for all } n \in \mathbb{N}$$
to find $\lim_{n\to\infty} \frac{\ln(n)}{n}$. Diving the given inequalities by n (which is positive) gives $0 < \frac{\ln(n)}{n}$ and $\frac{\ln(n)}{n} < \frac{\sqrt{n}}{n}$. Using basic algebra,
$$\frac{\sqrt{n}}{n} = \frac{n^{1/2}}{n} = n^{-1/2} = \left(\frac{1}{n}\right)^{1/2},$$
so $\lim_{n\to\infty} \frac{\sqrt{n}}{n} = \left(\lim_{n\to\infty} \frac{1}{n}\right)^{1/2} = 0$, and the Squeeze Theorem gives $\lim_{n\to\infty} \frac{\ln(n)}{n} = \boxed{0}$.

36. Use the Squeeze Theorem to determine the value of $\lim_{n\to\infty} \left(5^n + 3^n\right)^{1/n}$. $\boxed{5}$ from
$$\left(5^n\right)^{1/n} \leq \left(5^n + 3^n\right)^{1/n} \leq \left(5^n + 5^n\right)^{1/n}.$$

37. Evaluate $\lim_{n\to\infty} \frac{n^3}{3^n}$. $\boxed{0}$

38. Find the limits of these sequences and functions:
   (a) $\lim_{n\to\infty} \frac{2^n + 4^{n+1/2}}{4^n} = \boxed{2}$
   (b) $\lim_{x\to\infty} \frac{2^x + 4^{x+1/2}}{4^x} = \boxed{2}$
   (c) $\lim_{n\to\infty} \frac{n^3 + n^{-3}}{n^2 + n^{-9}} = \boxed{\infty}$
   (d) $\lim_{x\to\infty} \frac{x^3 + x^{-3}}{x^2 + x^{-9}} = \boxed{\infty}$
   (e) $\lim_{n\to\infty} \sin(\pi n) = \boxed{0}$ because $\sin(\pi n) = 0$ for all $n \in \mathbb{N}$
   (f) $\lim_{x\to\infty} \sin(\pi x)$ $\boxed{\text{doesn't exist}}$

39. Calculate $\lim_{x\to\infty} 6^x = \boxed{\infty}$ and $\lim_{x\to-\infty} 6^x = \boxed{0}$.

If $\lim_{x\to a} f(x)$ exists, then $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ both exist and are equal.
If $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ have different values, or at least one of them does not exist, then $\lim_{x\to a} f(x)$ does not exist.

40. Fill in the following table, then determine whether $\lim_{x \to -7} \frac{2x + 16}{1 - x}$ exists. If it exists, what is its value?

x	-7.1	-7.08	-7.003	-7.0001	-6.9999	-6.998	-6.96
$f(x)$	0.22222	0.22772	0.24916	0.24997	0.25003	0.25056	0.26131

$$\lim_{x \to -7} \frac{2x + 16}{1 - x} = \boxed{0.25 = \frac{1}{4}}$$

41. For the function $f(x) = \begin{cases} \sqrt{x} & \text{if } x \leq 4 \\ x^2 & \text{if } x > 4 \end{cases}$

(a) Fill in the following table, then determine whether $\lim_{x \to 4^-} f(x)$ (also written $\lim_{x \nearrow 4} f(x)$ or $\lim_{x \uparrow 4} f(x)$ in some books) exists. If it exists, what is its value?

x	3.9	3.95	3.975	3.9999
$f(x)$	1.9748	1.98746	1.99374	1.99997

$\lim_{x \to 4^-} f(x) = \boxed{5}$

(b) Fill in the following table, then determine whether $\lim_{x \to 4^+} f(x)$ (also written $\lim_{x \searrow 4} f(x)$ or $\lim_{x \downarrow 4} f(x)$ in some books) exists. If it exists, what is its value?

x	4.5	4.25	4.1	4.001	4.00006
$f(x)$	20.25	18.0625	16.81	16.008	16.0048

$\lim_{x \to 4^+} f(x) = \boxed{16}$

(c) Does $\lim_{x \to 4} f(x)$ exist? If it exists, what is its value?

The limit $\boxed{\text{does not exist}}$ because $\lim_{x \to 4^-} f(x) \neq \lim_{x \to 4^+} f(x)$.

42. For the function whose graph is shown below, give the following limits (if they exist) to the nearest 0.5.

(a) $\lim_{x \to 1^-} f(x)$ $\boxed{2}$

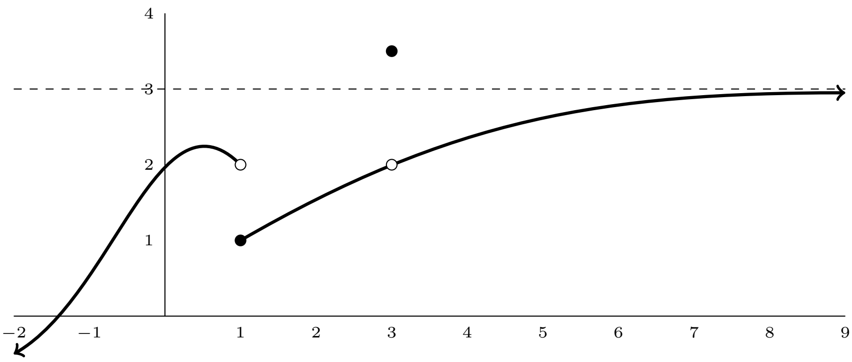
(b) $\lim_{x \to 1^+} f(x)$ $\boxed{1}$

(c) $\lim_{x \to 1} f(x)$ $\boxed{\text{does not exist}}$

(d) $\lim_{x \to 2} f(x)$ $\boxed{1.5}$ (or something similar)

(e) $\lim_{x \to 3} f(x)$ $\boxed{2}$ (*not* 3.5, although $f(3) = 3.5$)

(f) $\lim_{x \to \infty} f(x)$ $\boxed{3}$



43. Determine whether $\lim_{x\to 0^+} \frac{|x|}{x}$ exists. If it exists, what is its value? $\boxed{1}$

44. Determine whether $\lim_{x\to 0} \frac{|x|}{x}$ exists. If it exists, what is its value? $\boxed{\text{does not exist}}$

45. (a) Which of the functions below satisfy $\lim_{x\to 0^+} f(x) = 0$? x^2, $x^{1/2}$, $\sin(x)$, $\tan(x)$

(b) Which of the functions below satisfy $\lim_{x\to 0^+} f(x) = -\infty$? $\ln(x)$

x^2, x^{-2}, $x^{1/2}$, 2^x, $\ln(x)$, $\sin(x)$, $\cos(x)$, $\tan(x)$

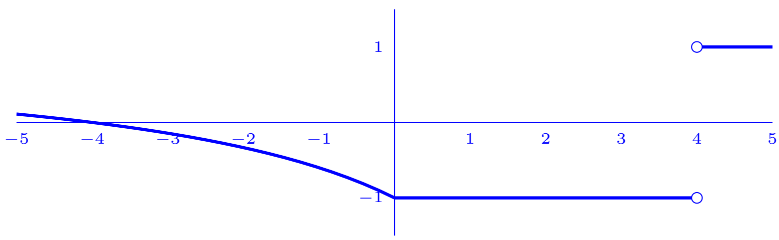
46. Does $\lim_{x\to 0} \frac{|x|-4}{|x-4|}$ exist? $\boxed{\text{Yes}}$ Does $\lim_{x\to 4} \frac{|x|-4}{|x-4|}$ exist? $\boxed{\text{No}}$ Draw a graph of the function for x-values between -5 and 5.

At $x = 0$, $f = \frac{0-4}{4} = -1$. At $x = 4$ the function is not defined. There are three regions to consider:

- $x > 4$ (in which $|x| = x$ and $|x-4| = x-4$),
- $0 < x < 4$ (in which $|x| = x$ but $|x-4| = 4-x$),
- $x < 0$ (in which $|x| = -x$ and $|x-4| = 4-x$)

In fact, we can write this as a piecewise function:

$$\frac{|x|-4}{|x-4|} = \begin{cases} \frac{-x-4}{4-x} & \text{if } x < 0 \\ \frac{x-4}{4-x} & \text{if } 0 \le x < 4 \\ \frac{x-4}{x-4} & \text{if } x > 4 \end{cases} = \begin{cases} \frac{x+4}{x-4} & \text{if } x < 0 \\ -1 & \text{if } 0 \le x < 4 \\ 1 & \text{if } x > 4 \end{cases}$$



47. Using the function $g(x) = \begin{cases} x^2 & \text{if } x \le -2 \\ x & \text{if } -2 < x < 2, \\ 4 & \text{if } x = 2 \\ 3^{-x} & \text{if } x > 2 \end{cases}$ calculate the following:

(a) $\lim_{x\to-\infty} g(x) = \lim_{x\to-\infty} x^2 = \boxed{+\infty}$, or you can say it does not exist

(b) $\lim_{x\to(-2)^-} g(x) = \lim_{x\to-2^-} x^2 = \boxed{4}$

(c) $\lim_{x\to(-2)^+} g(x) = \lim_{x\to-2^+} x = \boxed{-2}$

(d) $\lim_{x\to-2} g(x)$ does not exist because $4 \neq -2$

(e) $\lim_{x\to 2^-} g(x) = \lim_{x\to 2^-} x = \boxed{2}$

(f) $\lim_{x\to\infty} g(x) = \lim_{x\to\infty} 3^{-x} = \boxed{0}$

48. Calculate $\lim_{t\to 8} \frac{t+4+t^{1/3}}{t^2-8t+7}$. Just plug in $t=8$! $\frac{8+4+2}{64-64+7} = \frac{14}{7} = \boxed{2}$.

49. Calculate $\lim_{t\to-3} \frac{\sqrt{2t+22}-4}{t+3}$.

$$\lim_{t\to-3} \frac{\sqrt{2t+22}-4}{t+3} = \lim_{t\to-3} \frac{\left(\sqrt{2t+22}-4\right)}{(t+3)} \frac{\left(\sqrt{2t+22}+4\right)}{\left(\sqrt{2t+22}+4\right)} = \lim_{t\to-3} \frac{2t+22-16}{(t+3)\left(\sqrt{2t+22}+4\right)}$$

$$= \lim_{t\to-3} \frac{2\,(t+3)}{(t+3)\left(\sqrt{2t+22}+4\right)} = \lim_{t\to-3} \frac{2}{\sqrt{2t+22}+4} = \frac{2}{8} = \boxed{\frac{1}{4}}$$

50. (a) Expand $(\sqrt{h+1}-1)(\sqrt{h+1}+1)$ and then simplify as much as possible.

$(\sqrt{h+1}-1)(\sqrt{h+1}+1) = (h+1)^2 - \sqrt{h+1} - \sqrt{h+1} - 1 = \boxed{h}$

(b) Calculate $\lim_{h\to 0} \frac{\sqrt{h+1}-1}{h}$.

$$\lim_{h\to 0} \frac{\sqrt{h+1}-1}{h} \cdot \frac{\sqrt{h+1}+1}{\sqrt{h+1}+1} = \lim_{h\to 0} \frac{h}{h\sqrt{h+1}+h} = \lim_{h\to 0} \frac{1}{\sqrt{h+1}+1} = \boxed{\frac{1}{2}}$$

51. Find all value(s) of p for which $\lim_{x\to 8} f(x)$ exists if

$$f(x) = \begin{cases} 3x+p & \text{if } x \leq 8 \\ 2x-5 & \text{if } x > 8. \end{cases} \quad \boxed{p=-13}$$

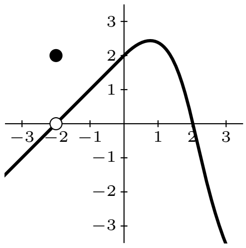
52. (a) Find $\lim_{x\to 0} \frac{(5+x)^3-125}{x} = \boxed{75}$

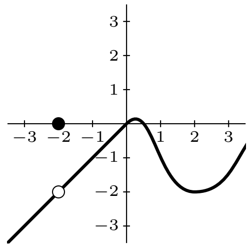
(b) Find $\lim_{h\to 0} \frac{(5+h)^3-125}{h} = \boxed{75}$

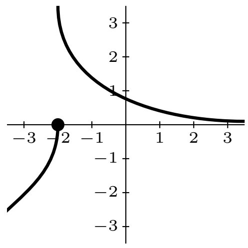
(c) Find $\lim_{h\to 0} \frac{(x+h)^3-x^3}{h}$. Your answer will be a formula with x. $\boxed{3x^2}$

☆53. Find $\lim_{x\to 0} (1+tx)^{1/x}$. Your answer will be a formula with t. $\boxed{e^t}$

54. For each graph $y = f(x)$ below, is $\lim_{x \to -2^+} f(x) = 0$ true? $\boxed{\text{(a) Yes, (b) No, (c) No}}$

(a) 

(b) 

(c) 

55. For each graph $y = f(x)$ from Task **??**, does $\lim_{x \to -2} f(x)$ exist?

$\boxed{\text{(a) Yes, (b) Yes, (c) No}}$

A function $f(x)$ is **continuous at $x = p$** if $f(p)$ and $\lim_{x \to p} f(x)$ both exist and are equal to each other. If not, then $f(x)$ is **discontinuous at $x = p$**.

A "jump", "hole", or "vertical asymptote" in a graph $y = f(x)$ will cause $f(x)$ to be discontinuous.

56. For each graph $y = f(x)$ from Task **??**, is $f(x)$ continuous at $x = 2$? $\boxed{\text{all "No"}}$

57. Give the following limits:

(a) $\lim_{x \to (\pi/4)^-} \tan(x) = \boxed{1}$

(b) $\lim_{x \to (\pi/4)^+} \tan(x) = \boxed{1}$

(c) $\lim_{x \to (\pi/2)^-} \tan(x) = \boxed{+\infty}$

(d) $\lim_{x \to (\pi/2)^+} \tan(x) = \boxed{-\infty}$

58. (a) Find the vertical asymptote(s) of

$$g(x) = \frac{1}{x^2 + x - 6}. \qquad \boxed{x = -3, x = 2}$$

(b) Find the vertical asymptote(s) of

$$f(x) = \frac{x^2 - x - 2}{x^2 + x - 6}. \qquad \boxed{x = -3 \text{ only}}$$

59. What horizontal asymptotes does the function

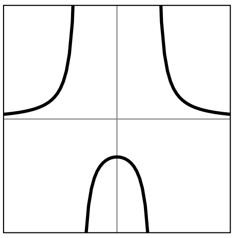
$$f(x) = \frac{x}{|x| + 5}$$

have? Hint: Calculate $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$. $\boxed{y = 1, y = -1}$

60. Match the functions with their graphs:

(a) $\dfrac{x}{x^2-1}$ $\boxed{\text{(II)}}$ (b) $\dfrac{1}{x^2-1}$ $\boxed{\text{(I)}}$ (c) $\dfrac{x+1}{x^2-1}$ $\boxed{\text{(IV)}}$ (d) $\dfrac{x^2}{x^2-1}$ $\boxed{\text{(III)}}$

(I)



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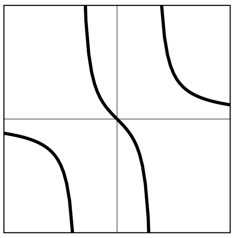

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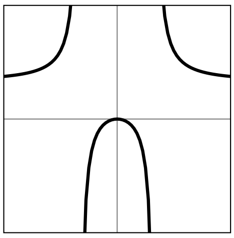

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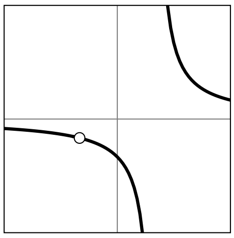


61. Calculate $\lim_{x\to 0} x^2 \cos\left(\frac{1}{x}\right)$ using the Squeeze Theorem for functions.

$$-1 \le \cos(1/x) \le 1$$
$$-x^2 \le x^2\cos(1/x) \le x^2$$
$$\lim_{x\to 0} -x^2 \le \lim_{x\to 0} x^2\cos(1/x) \le \lim_{x\to 0} x^2$$
$$0 \le \lim_{x\to 0} x^2\cos(1/x) \le 0$$
$$x^2\cos(1/x) = \boxed{0}$$

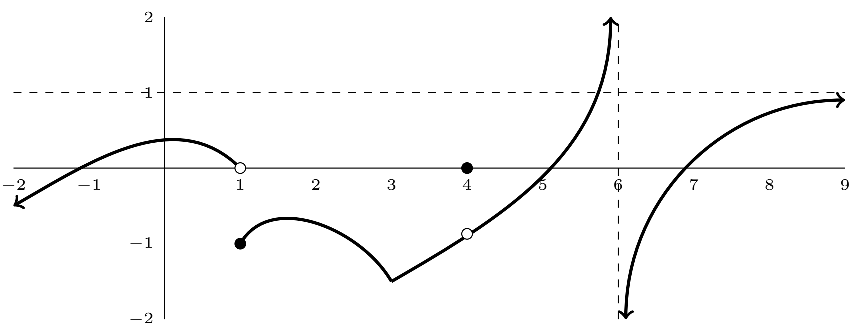
62. If $f(x)$ is a function for which

$$24x - 41 \quad \le \quad f(x) \quad \le \quad 4x^2 - 5$$

for all x, what is $\lim_{x\to 3} f(x)$?

$\lim_{x\to 3}(24x-41) = 31$ and $\lim_{x\to 3}(4x^2-5) = 31$, so the Squeeze Theorem guarantees that $\lim_{x\to 3} f(x) = \boxed{31}$.

63. List all points where the function graphed below is discontinuous.



$\boxed{x = 1, x = 4, x = 6}$ The function *is* continuous at $x = 3$.

64. Give an example of a function that is discontinuous at infinitely many points.

There are many examples. Here are two:

- $\boxed{\tan(x)}$ is discontinuous (in fact, undefined) at all $x = \dfrac{\pm\pi}{2} + 2\pi n$ for integer n.

- The "floor" function $\lfloor x \rfloor$ is discontinuous at every integer $x = n$.

☆65. Give an example of a function that is discontinuous at *every* point.

The "Dirichlet function" is a famous (well, famous within mathematics) example: $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$

66. For what value(s) of p is the function

$$f(x) = \begin{cases} x^3 + 5 & \text{if } x < -2 \\ x + p & \text{if } x \geq -2 \end{cases}$$

continuous?

$\lim_{x \to (-2)^-} f(x) = (-2)^3 + 5 = -3$, and $\lim_{x \to (-2)^+} f(x) = (-2) + p$. If f is continuous then we need $-3 = -2 + p$, which means $\boxed{p = -1}$.

67. Which of the following functions has a hole at $x = 8$? $\boxed{\text{(C)}}$

(B) $\dfrac{x^2 - 8x - 9}{x^2 + 8x + 7}$ hole at $x = -1$

(A) $\dfrac{x^2 - 8x - 9}{x^2 - 7x - 8}$ asympt. at $x = 8$

$\boxed{\text{(C)}}$ $\dfrac{x^2 - 9x + 8}{x^2 - 7x - 8}$

68. Is $\dfrac{5x^2 + 1}{x^2 - 1}$ continuous? $\boxed{\text{no}}$ at $x = -1$ and at $x = 1$ Is $\dfrac{5x^2 + 1}{x^2 + 1}$? $\boxed{\text{yes}}$

69. Without graphing, determine which one of the three equations below has a solution with $0 \leq x \leq 3$.

(A) $x^2 = 4^x$, (B) $x^3 = 5^x$, (C) $x^5 = 6^x$.

$\boxed{\text{(C)}}$ because the function $f(x) = x^5 - 6^x$ has $f(0) = -1$ and $f(3) = 27$. Since $-1 < 0 < 27$, by the Intermediate Value Theorem, there must exist an x in $[0, 3]$ such that $f(x) = 0$.

70. Let $f(x) = \dfrac{13x - 77}{x - 5}$.

(a) $f(4) = 25$ and $f(11) = 11$. Does the Intermediate Value Theorem guarantee that $f(x) = 10$ for some $x \in [4, 11]$?
$\boxed{\text{No}}$ because f is discontinuous at $x = 5$.

(b) $f(6) = 1$ and $f(11) = 11$. Does the Intermediate Value Theorem guarantee that $f(x) = 10$ for some $x \in [6, 11]$?
$\boxed{\text{Yes}}$ because f is continuous on $[6, 11]$. (In fact $f(9) = 10$, though the task does not ask for this.)

(c) $f(6) = 1$ and $f(8) = 9$. Does the Intermediate Value Theorem guarantee that $f(x) = 10$ for some $x \in [6, 8]$?
$\boxed{\text{No}}$ because 10 is not in the y-interval $[f(6), f(8)] = [1, 9]$.

71. Label each of the following expressions as “a sum”, “a difference”, “a product”, “a quotient”, or “a composition”.

   (a) $x^2 + 7$ sum or composition

   (b) $(x + 7)^2$ composition or product

   (c) $\sin(x + 7)$ composition

   (d) $\dfrac{(x-1)^3}{e^x} - \dfrac{1}{x+8}$ difference

   (e) $\dfrac{5\sin(2x)}{e^{(\sin(x))^3}}$ quotient

   (f) $\sqrt{\dfrac{1}{x} + \dfrac{1}{x^2}}$ composition

   (g) $\sin(\sqrt{x}) + \sqrt[3]{\sin(x)}$ sum

72. Give the composition $f \circ g$ for the functions $f(x) = e^x$ and $g(x) = 8x - 3$. e^{8x-3} Also give $g \circ f$. $8e^x - 3$

☆73. Use the definition of a limit with ε and δ to show that the limit of

$$f(x) = 4x - 3$$

as x approaches 2 is equal to 5.

As a reminder, starred ☆ tasks are ones that I (Adam) believe are beyond the level of an introductory calculus class.

Let $\varepsilon > 0$ be any positive value. We need to find some $\delta > 0$ such that

$$\text{If } 0 < \left|x - 2\right| < \delta \text{ then } \left|(4x - 3) - 5\right| < \varepsilon.$$

Let $\delta = \dfrac{\varepsilon}{4}$. Because $\varepsilon > 0$, we have $\delta > 0$ also. If $\left|x - 2\right| < \delta$ then

$$\begin{aligned} -\delta < {} & x - 2 < \delta \\ -\varepsilon/4 < {} & x - 2 < \varepsilon/4 \\ -\varepsilon < {} & 4x - 8 < \varepsilon \\ -\varepsilon < {} & (4x - 3) - 5 < \varepsilon \end{aligned}$$

and thus $\left|(4x - 3) - 5\right| < \varepsilon$.