Analysis 1, Summer 2023

# List 2

*Limits of functions*

35. Use the facts

$$0 < \ln(n) \qquad \text{for all } n \in \mathbb{N} \text{ with } n \geq 2$$

and

$$\ln(n) < \sqrt{n} \qquad \text{for all } n \in \mathbb{N}$$

to find $\lim\limits_{n\to\infty} \dfrac{\ln(n)}{n}$.

36. Use the Squeeze Theorem to determine the value of $\lim\limits_{n\to\infty} \left(5^n + 3^n\right)^{1/n}$.

37. Evaluate $\lim\limits_{n\to\infty} \dfrac{n^3}{3^n}$.

38. Find the limits of these sequences and functions:

(a) $\lim\limits_{n\to\infty} \dfrac{2^n + 4^{n+1/2}}{4^n}$

(b) $\lim\limits_{x\to\infty} \dfrac{2^x + 4^{x+1/2}}{4^x}$

(c) $\lim\limits_{n\to\infty} \dfrac{n^3 + n^{-3}}{n^2 + n^{-9}}$

(d) $\lim\limits_{x\to\infty} \dfrac{x^3 + x^{-3}}{x^2 + x^{-9}}$

(e) $\lim\limits_{n\to\infty} \sin(\pi n)$

(f) $\lim\limits_{x\to\infty} \sin(\pi x)$

39. Calculate $\lim\limits_{x\to\infty} 6^x$ and $\lim\limits_{x\to-\infty} 6^x$.

If $\lim\limits_{x\to a} f(x)$ exists, then $\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to a^+} f(x)$ both exist and are equal.
If $\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to a^+} f(x)$ have different values, or at least one of them does not exist, then $\lim\limits_{x\to a} f(x)$ does not exist.

40. Fill in the following table, then determine whether $\lim\limits_{x\to-7} \dfrac{2x+16}{1-x}$ exists. If it exists, what is its value?

| $x$ | $-7.1$ | $-7.08$ | $-7.003$ | $-7.0001$ | $-6.9999$ | $-6.998$ | $-6.96$ |
|---|---|---|---|---|---|---|---|
| $f(x)$ | | | | | | | |

41. For the function $f(x) = \begin{cases} \sqrt{x} & \text{if } x \leq 4 \\ x^2 & \text{if } x > 4 \end{cases}$

(a) Fill in the following table, then determine whether $\lim\limits_{x\to 4^-} f(x)$ (also written $\lim\limits_{x\nearrow 4} f(x)$ or $\lim\limits_{x\uparrow 4} f(x)$ in some books) exists. If it exists, what is its value?

| $x$ | $3.9$ | $3.95$ | $3.975$ | $3.9999$ |
|---|---|---|---|---|
| $f(x)$ | | | | |

(b) Fill in the following table, then determine whether $\lim\limits_{x\to 4^+} f(x)$ (also written $\lim\limits_{x\searrow 4} f(x)$ or $\lim\limits_{x\downarrow 4} f(x)$ in some books) exists. If it exists, what is its value?

| $x$ | $4.5$ | $4.25$ | $4.1$ | $4.001$ | $4.00006$ |
|---|---|---|---|---|---|
| $f(x)$ | | | | | |

(c) Does $\lim\limits_{x\to 4} f(x)$ exist? If it exists, what is its value?

42. For the function whose graph is shown below, give the following limits (if they exist) to the nearest 0.5.

(a) $\lim_{x\to 1^-} f(x)$

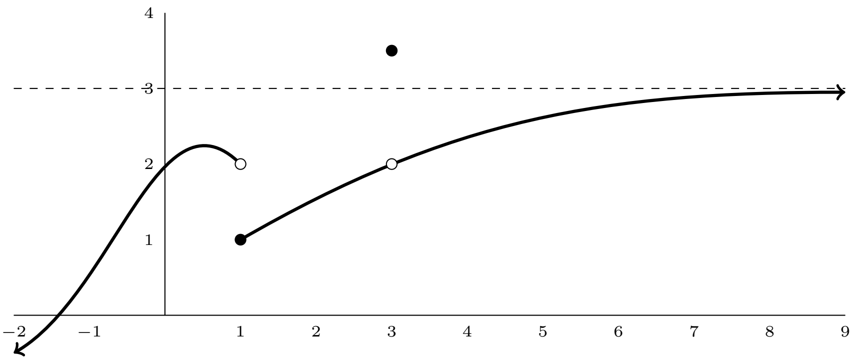
(b) $\lim_{x\to 1^+} f(x)$

(c) $\lim_{x\to 1} f(x)$

(d) $\lim_{x\to 2} f(x)$

(e) $\lim_{x\to 3} f(x)$

(f) $\lim_{x\to \infty} f(x)$




43. Determine whether $\lim_{x\to 0^+} \frac{|x|}{x}$ exists. If it exists, what is its value?

44. Determine whether $\lim_{x\to 0} \frac{|x|}{x}$ exists. If it exists, what is its value?

45. (a) Which of the functions below satisfy $\lim_{x\to 0^+} f(x) = 0$?

(b) Which of the functions below satisfy $\lim_{x\to 0^+} f(x) = -\infty$?

$x^2$, $x^{-2}$, $x^{1/2}$, $2^x$, $\ln(x)$, $\sin(x)$, $\cos(x)$, $\tan(x)$

46. Does $\lim_{x\to 0} \frac{|x|-4}{|x-4|}$ exist? Does $\lim_{x\to 4} \frac{|x|-4}{|x-4|}$ exist? Draw a graph of the function for $x$-values between $-5$ and $5$.

47. Using the function $g(x) = \begin{cases} x^2 & \text{if } x \le -2 \\ x & \text{if } -2 < x < 2, \\ 4 & \text{if } x = 2 \\ 3^{-x} & \text{if } x > 2 \end{cases}$ calculate the following:

(a) $\lim_{x\to -\infty} g(x)$

(b) $\lim_{x\to (-2)^-} g(x)$

(c) $\lim_{x\to (-2)^+} g(x)$

(d) $\lim_{x\to -2} g(x)$

(e) $\lim_{x\to 2^-} g(x)$

(f) $\lim_{x\to \infty} g(x)$

48. Calculate $\lim_{t\to 8} \frac{t+4+t^{1/3}}{t^2-8t+7}$.

49. Calculate $\lim_{t\to -3} \frac{\sqrt{2t+22}-4}{t+3}$.

50. (a) Expand $(\sqrt{h+1}-1)(\sqrt{h+1}+1)$ and then simplify as much as possible.

(b) Calculate $\displaystyle\lim_{h\to 0} \frac{\sqrt{h+1}-1}{h}$.

51. Find all value(s) of $p$ for which $\lim_{x\to 8} f(x)$ exists if

$$f(x) = \begin{cases} 3x+p & \text{if } x \le 8 \\ 2x-5 & \text{if } x > 8. \end{cases}$$

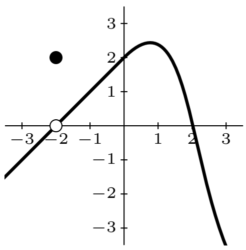
52. (a) Find $\displaystyle\lim_{x\to 0} \frac{(5+x)^3-125}{x}$.

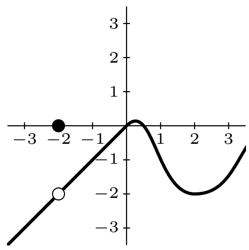
(b) Find $\displaystyle\lim_{h\to 0} \frac{(5+h)^3-125}{h}$.

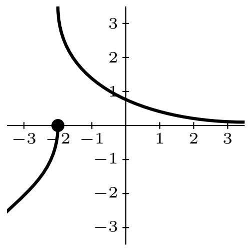
(c) Find $\displaystyle\lim_{h\to 0} \frac{(x+h)^3-x^3}{h}$. Your answer will be a formula with $x$.

☆53. Find $\lim_{x\to 0} (1+tx)^{1/x}$. Your answer will be a formula with $t$.

54. For each graph $y = f(x)$ below, is $\lim_{x\to -2^+} f(x) = 0$ true?

(a) 


(b) 


(c) 


55. For each graph $y = f(x)$ from Task 54, does $\lim_{x\to -2} f(x)$ exist?

A function $f(x)$ is **continuous at $x = p$** if $f(p)$ and $\lim_{x\to p} f(x)$ both exist and are equal to each other. If not, then $f(x)$ is **discontinuous at $x = p$**.

A "jump", "hole", or "vertical asymptote" in a graph $y = f(x)$ will cause $f(x)$ to be discontinuous.

56. For each graph $y = f(x)$ from Task 54, is $f(x)$ continuous at $x = 2$?

57. Give the following limits:

(a) $\displaystyle\lim_{x\to(\pi/4)^-} \tan(x)$

(b) $\displaystyle\lim_{x\to(\pi/4)^+} \tan(x)$

(c) $\displaystyle\lim_{x\to(\pi/2)^-} \tan(x)$

(d) $\displaystyle\lim_{x\to(\pi/2)^+} \tan(x)$

58. (a) Find the vertical asymptote(s) of

$$g(x) = \frac{1}{x^2+x-6}.$$

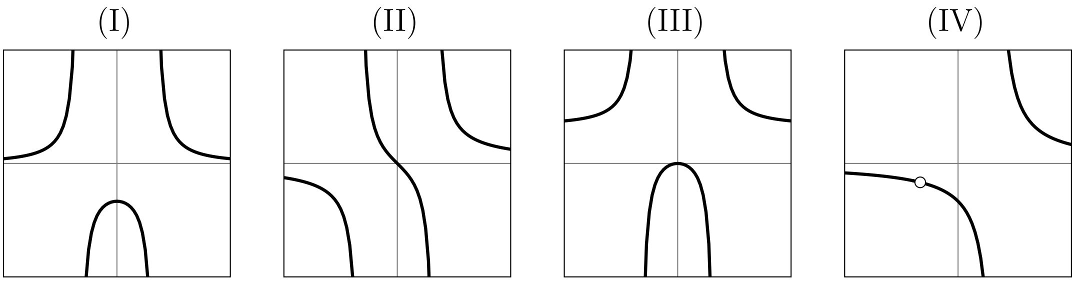
(b) Find the vertical asymptote(s) of

$$f(x) = \frac{x^2-x-2}{x^2+x-6}.$$

59. What horizontal asymptotes does the function
$$f(x) = \frac{x}{|x| + 5}$$
have? Hint: Calculate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

60. Match the functions with their graphs:

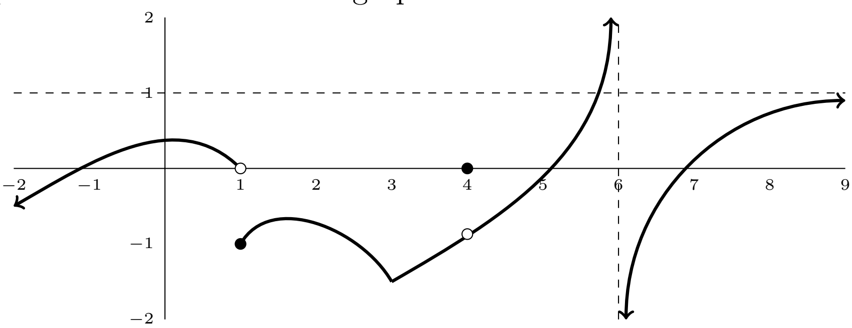
(a) $\frac{x}{x^2-1}$ (b) $\frac{1}{x^2-1}$ (c) $\frac{x+1}{x^2-1}$ (d) $\frac{x^2}{x^2-1}$




61. Calculate $\lim_{x\to 0} x^2 \cos\left(\frac{1}{x}\right)$ using the Squeeze Theorem for functions.

62. If $f(x)$ is a function for which
$$24x - 41 \quad\leq\quad f(x) \quad\leq\quad 4x^2 - 5$$
for all $x$, what is $\lim_{x\to 3} f(x)$?

63. List all points where the function graphed below is discontinuous.




64. Give an example of a function that is discontinuous at infinitely many points.

☆65. Give an example of a function that is discontinuous at *every* point.

66. For what value(s) of $p$ is the function
$$f(x) = \begin{cases} x^3 + 5 & \text{if } x < -2 \\ x + p & \text{if } x \geq -2 \end{cases}$$
continuous?

67. Which of the following functions has a hole at $x = 8$?

(B) $\frac{x^2 - 8x - 9}{x^2 + 8x + 7}$ (A) $\frac{x^2 - 8x - 9}{x^2 - 7x - 8}$ (C) $\frac{x^2 - 9x + 8}{x^2 - 7x - 8}$

68. Is $\frac{5x^2+1}{x^2-1}$ continuous? Is $\frac{5x^2+1}{x^2+1}$?

69. Without graphing, determine which one of the three equations below has a solution with $0 \le x \le 3$.

(A) $x^2 = 4^x$, (B) $x^3 = 5^x$, (C) $x^5 = 6^x$.

70. Let $f(x) = \dfrac{13x - 77}{x - 5}$.

(a) $f(4) = 25$ and $f(11) = 11$. Does the Intermediate Value Theorem guarantee that $f(x) = 10$ for some $x \in [4, 11]$?

(b) $f(6) = 1$ and $f(11) = 11$. Does the Intermediate Value Theorem guarantee that $f(x) = 10$ for some $x \in [6, 11]$?

(c) $f(6) = 1$ and $f(8) = 9$. Does the Intermediate Value Theorem guarantee that $f(x) = 10$ for some $x \in [6, 8]$?

71. Label each of the following expressions as "a sum", "a difference", "a product", "a quotient", or "a composition".

(a) $x^2 + 7$

(b) $(x + 7)^2$

(c) $\sin(x + 7)$

(d) $\dfrac{(x-1)^3}{e^x} - \dfrac{1}{x+8}$

(e) $\dfrac{5\sin(2x)}{e^{(\sin(x))^3}}$

(f) $\sqrt{\dfrac{1}{x} + \dfrac{1}{x^2}}$

(g) $\sin(\sqrt{x}) + \sqrt[3]{\sin(x)}$

72. Give the composition $f \circ g$ for the functions $f(x) = e^x$ and $g(x) = 8x - 3$.

☆73. Use the definition of a limit with $\varepsilon$ and $\delta$ to show that the limit of

$$f(x) = 4x - 3$$

as $x$ approaches 2 is equal to 5.

**As a reminder, starred ☆ tasks are ones that I (Adam) believe are beyond the level of an introductory calculus class.**