Analysis 1, Summer 2023

List 3

Derivatives

☆73. Use the (ε, δ) definition of a limit to show that the limit of

$$f(x) = 4x - 3$$

as x approaches 2 is equal to 5.

As a reminder, starred ☆ tasks are ones that I (Adam) believe are beyond the level of an introductory calculus class.

Let $\varepsilon > 0$ be any positive value. We need to find some $\delta > 0$ such that

$$\text{If } 0 < |x-2| < \delta \text{ then } |(4x-3)-5| < \varepsilon.$$

Let $\delta = \frac{\varepsilon}{4}$. Because $\varepsilon > 0$, we have $\delta > 0$ also. If $|x-2| < \delta$ then

$$\begin{aligned} -\delta < {} & x-2 && < \delta \\ -\varepsilon/4 < {} & x-2 && < \varepsilon/4 \\ -\varepsilon < {} & 4x-8 && < \varepsilon \\ -\varepsilon < {} & (4x-3)-5 && < \varepsilon \end{aligned}$$

and thus $|(4x-3)-5| < \varepsilon$.

74. Use the limit definition of a derivative (below) to show that the derivative of

$$f(x) = \frac{36}{x+1}$$

at $x = 2$ is equal to -4. **This task is *not* starred.**

$$\begin{aligned} f'(2) &= \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} \\ &= \lim_{h\to 0} \frac{\frac{36}{2+h+1} - \frac{36}{2+1}}{h} = \lim_{h\to 0} \frac{\frac{36}{3+h} - 12}{h} = \lim_{h\to 0} \frac{\frac{36}{3+h} - \frac{12(3+h)}{3+h}}{h} \\ &= \lim_{h\to 0} \frac{\frac{36-12(3+h)}{3+h}}{h} = \lim_{h\to 0} \frac{36-36-12h}{3h+h^2} = \lim_{h\to 0} \frac{-12h}{3h+h^2} = \lim_{h\to 0} \frac{-12}{3+h} \\ &= \frac{-12}{3+(0)} = \frac{-12}{3} = -4 \end{aligned}$$

75. Without graphing, determine which one of the three equations below has a solution with $0 \le x \le 3$.

(A) $x^2 = 4^x$ (B) $x^3 = 5^x$ (C) $x^5 = 6^x$

$\boxed{\text{(C)}}$ because the function $f(x) = x^5 - 6^x$ has $f(0) = -1$ and $f(3) = 27$. Since $-1 < 0 < 27$, by the Intermediate Value Theorem, there must exist an x in $[0, 3]$ such that $f(x) = 0$.

For a function $f(x)$ and a number a, the **derivative of f at a**, written $f'(a)$, is the slope of the tangent line to $y = f(x)$ at the point $(a, f(a))$ and is calculated as

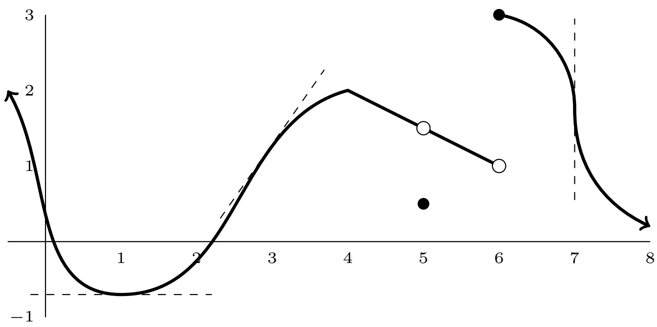
$$f'(a) = \lim_{\Delta x\to 0} \frac{\Delta f}{\Delta x} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}.$$

The function $f(x)$ is **differentiable at a** if $f'(a)$ exists and is finite.

76. Calculate $f'(5)$ for the function $f(x) = x^3$. Hint: See Task 52(b). $\boxed{75}$

77. Calculate $f'(1)$ for the function $f(x) = \sqrt{x}$. Hint: See Task 50(b). $\boxed{\frac{1}{2}}$

78. The graph of a function is shown below. Near $x = 1$, $x = 3$, and $x = 7$, part of the tangent lines to the graph at those points is shown as a dashed line segment.



(a) List all points where the function is not continuous. $\boxed{x = 5, x = 6}$

(b) List all points where the function is not differentiable (that is, where the derivative does not exist). $\boxed{x = 4, x = 5, x = 6, x = 7}$

79. List all points where $f(x) = \dfrac{|x| - 4}{|x - 4|}$ is not differentiable. $\boxed{x = 0 \text{ and } x = 4}$ The function has a "corner" at $x = 0$, so it is continuous there but not differentiable. It has a "jump" at $x = 4$, so it is not continuous and also not differentiable there.

80. (a) If $S(x) = f(x) + g(x)$, does that mean that $S'(3) = f'(3) + g'(3)$? That is, is

$$\lim_{h\to 0} \frac{\big(f(3+h) + g(3+h)\big) - \big(f(3) + g(3)\big)}{h} = \lim_{h\to 0} \frac{f(3+h) - f(3)}{h} + \lim_{h\to 0} \frac{g(3+h) - g(3)}{h}$$

always true? $\boxed{\text{Yes}}$

(b) If $P(x) = f(x) \cdot g(x)$, does that mean that $P'(3) = f'(3) \cdot g'(3)$? That is, is

$$\lim_{h\to 0} \frac{\big(f(3+h) \cdot g(3+h)\big) - \big(f(3) \cdot g(3)\big)}{h} = \lim_{h\to 0} \frac{f(3+h) - f(3)}{h} \cdot \lim_{h\to 0} \frac{g(3+h) - g(3)}{h}$$

always true? $\boxed{\text{No}}$

The **linear approximation to $f(x)$ near $x = a$** is the function

$$L(x) = f(a) + f'(a)(x - a).$$

The line $y = L(x)$ is the **tangent line** to $y = f(x)$ at the point $(a, f(a))$.

81. Graph the curve $y = \sqrt{x}$ and the line tangent to that curve at $(1, 1)$.

82. (a) Give the linear approximation to $\sqrt{x}$ near $x = 1$.

$\boxed{L(x) = 1 + \frac{1}{2}(x-1)}$ (The slope is from Task 77.)

(b) Use the approximation from part (a) to estimate $\sqrt{1.2}$.

$L(1.2) = 1 + \frac{1}{2}(1.2 - 1) = 1 + \frac{0.2}{2} = \boxed{1.1}$

(c) The true value of $\sqrt{1.2}$ is 1.09545..., so is $L(1.2)$ a good approximation?

This question asks for an opinion, so you could answer "yes" or "no". I (Adam) would say "yes" because is correct to three decimal places (1.10), and the percentage error is only $\frac{1.1-1.09545}{1.09545} = 0.004 = 0.4\%$.

(d) Use the approximation from part (a) to estimate $\sqrt{8}$.

$L(5) = 1 + \frac{1}{2}(8 - 1) = \boxed{\frac{9}{2} = 4.5}$

(e) The true value of $\sqrt{8}$ is 2.82843..., so is $L(8)$ a good approximation?

I would say "no". The input 8 is not close to $x = 1$, so it is not surprising that $L(8)$ is not close to $\sqrt{8}$.

83. If f is a function with $f(-4) = 2$ and $f'(-4) = \frac{1}{3}$, give the linear approximation to $f(x)$ near $x = -4$. $\boxed{L(x) = 2 + \frac{1}{3}(x+4)}$

84. If g is a function with $g(5) = 12$ and $g'(5) = 2$, use a linear approximation to estimate the value of $g(4.9)$. $L(4.9) = \boxed{11.8}$

85. Give an equation for the tangent line to $y = 4x^2 - x$ at $x = 2$.

$\boxed{y = 14 + 15(x-2)}$ or $\boxed{y = 15x - 16}$

86. Give an equation for the tangent line to $7x + 2$ through the point $(30, 212)$.

$\boxed{y = 30 + 7(x - 212)}$ or $\boxed{y = 7x + 2}$. (Since $y = 7x + 2$ is a straight line, the tangent line to it—at any point—is exactly itself.)

The Constant Multiple Rule: If c is a constant then

$$(c\,f)' = c\,f' \qquad \big(c\,f(x)\big)' = c\,f'(x) \qquad \frac{\mathrm{d}}{\mathrm{d}x}\big[c\,f\big] = c\,\frac{\mathrm{d}f}{\mathrm{d}x} \qquad D\big[c\,f\big] = c\,D[f]$$

(these are four ways of writing exact the same fact).

The Sum Rule: $\frac{\mathrm{d}}{\mathrm{d}x}\big[f+g\big] = \frac{\mathrm{d}}{\mathrm{d}x}[f] + \frac{\mathrm{d}}{\mathrm{d}x}[g]$.

The Power Rule: If p is a constant then $\frac{\mathrm{d}}{\mathrm{d}x}\big[x^p\big] = p\,x^{p-1}$.

87. All parts of this task have exactly the same answer! Answer: $\boxed{14x^6}$

(a) Find $f'(x)$ for the function $f(x) = 2x^7$.

(b) Give f' if $f = 2x^7$.

(c) Find y' for $y = 2x^7$.

(d) Compute $\frac{\mathrm{d}f}{\mathrm{d}x}$ for the function $f(x) = 2x^7$.

(e) Compute $\frac{\mathrm{d}y}{\mathrm{d}x}$ for $y = 2x^7$.

(f) Give the derivative of $2x^7$ with respect to x.

(g) Find the derivative of $2x^7$.

(h) Calculate $\frac{\mathrm{d}}{\mathrm{d}x}2x^7$. (i) Calculate $\left(2x^7\right)'$. (j) Calculate $D\left[2x^7\right]$.

(k) Differentiate $2x^7$ with respect to x.

(ℓ) Differentiate $2x^7$.

88. Differentiate $x^5 + \frac{2}{9}x^3 + \sqrt{3x} + \dfrac{x^{10}}{\sqrt{x}}$. $\boxed{5x^4 + \dfrac{2}{3}x^2 + \dfrac{\sqrt{3}}{2\sqrt{x}} + \dfrac{19}{2}x^{17/2}}$

89. Differentiate $\left(x + \sqrt{x}\right)^2$. $\boxed{2x + 3\sqrt{x} + 1}$ or $2\left(x + \sqrt{x}\right)\left(1 + \frac{1}{2\sqrt{x}}\right)$

☆90. Differentiate $\left(x + \sqrt{x}\right)^{100}$. $\boxed{100\left(x + \sqrt{x}\right)^{99}\left(1 + \frac{1}{2\sqrt{x}}\right)}$

91. For each of the functions below, can the Power Rule and/or Constant Multiple Rule (along with maybe some algebra) be used to find the derivative? If so, give the derivative.

(a) $2x^6$ $\boxed{\text{Yes: } 12x^5}$

(b) $2\sqrt{x}$ $\boxed{\text{Yes: } x^{-1/2}}$ or $\dfrac{1}{\sqrt{x}}$

(c) $\sqrt{5x}$ $\boxed{\text{Yes: } \frac{\sqrt{5}}{2}x^{-1/2}}$

(d) x^{π} $\boxed{\text{Yes: } \pi x^{\pi-1}}$

(e) $x^{\sin x}$ $\boxed{\text{No}}$

(f) $(\sin x)^x$ $\boxed{\text{No}}$

(g) e^x $\boxed{\text{No!}}$

(h) $\cos(5x)$ $\boxed{\text{No}}$

(i) $\sin(5\cos(x))$ $\boxed{\text{No}}$

(j) $e^{5\ln(x)}$ $\boxed{\text{Yes: } 5x^4}$

(k) $\dfrac{3}{x^6}$ $\boxed{\text{Yes: } -18x^{-7}}$

(ℓ) x^x $\boxed{\text{No!}}$

(m) $\ln(2+x)$ $\boxed{\text{No}}$

(n) $\ln(2x)$ $\boxed{\text{No}}$

(o) $\ln(2^x)$ $\boxed{\text{Yes: } \ln(2)}$

(p) $\ln(x^2)$ $\boxed{\text{No}}$

92. Is it possible to find the derivative of the following functions using the Power Rule, Constant Multiple Rule, and Sum Rule?

(a) $x+\ln(5e^x)$ This function equals $2x + \ln(5)$, so $\boxed{\text{Yes: } 2}$.

(b) $\dfrac{2x}{x+6}$ $\boxed{\text{No}}$

(c) $\dfrac{x+6}{2x}$ $\boxed{\text{Yes: } -3x^{-2}}$ or $\frac{-3}{x^2}$

(d) $\dfrac{x+\frac{1}{x}}{\sqrt{x}}$ $\boxed{\text{Yes: } \frac{1}{2}x^{-1/2} - \frac{3}{2}x^{-5/2}}$

93. Give an equation for the tangent line to $y = x^3 - x$ at $x = 2$. $\boxed{y = 6 + 11(x-2)}$ Other formats, such as $y = 11x - 16$, may also be correct.

☆94. Find a line that is tangent to both $y = x^2 + 20$ and $y = x^3$. $\boxed{y = 12x - 16}$ is tangent to $y = x^2 + 20$ at $x = 6$ and tangent to $y = x^3$ at $x = 2$.

95. Give the derivative of each of the following functions.

(a) x^{7215} $\boxed{7215x^{7214}}$

(b) $5x^{100} + 9x$ $\boxed{500x^{99} + 9}$

(c) $2x^3 - 6x^2 + 10x + 1$ $\boxed{6x^2 - 12x + 10}$

(d) $3\sqrt{x}$ $\boxed{\frac{3}{2}x^{-1/2}}$ or $\boxed{\dfrac{3}{2\sqrt{x}}}$

(e) $\sqrt[3]{x}$ $\boxed{\frac{1}{3}x^{-2/3}}$ or $\boxed{\dfrac{1}{3\sqrt[3]{x^2}}}$

(f) $\sqrt{x}^3$ $\boxed{\frac{2}{3}x^{-1/3}}$ or $\boxed{\dfrac{2}{3\sqrt[3]{x}}}$

(g) 31 $\boxed{0}$

(h) $x + \frac{1}{x}$ $\boxed{1 - x^{-2}}$ or $\boxed{1 - \dfrac{1}{x^2}}$

(i) $\sqrt{x} + \frac{1}{\sqrt{x}}$ $\boxed{\frac{1}{2}x^{-1/2} + \frac{-1}{2}x^{-3/2}}$ or $\boxed{\dfrac{1}{2\sqrt{x}} - \dfrac{1}{2\sqrt{x^3}}}$

(j) $(3x+7)^2$ $\boxed{18x + 42}$ or $\boxed{6(3x+7)}$

96. Give an example of a function whose derivative is...

(a) x^2 Any function that is $\boxed{\frac{1}{3}x^3 + \text{a constant}}$. This is often written $\frac{1}{3}x^3 + C$. Correct specific examples include $\frac{1}{3}x^3 + 5$ and $\frac{1}{3}x^3 - \frac{328}{101}$ and just $\frac{1}{3}x^3$.

(b) $\sqrt{x}$ The simplest answer is $\boxed{\frac{2}{3}x^{3/2}}$.

(c) $\dfrac{1}{x^2}$ The simplest answer is $\boxed{\dfrac{-1}{x}}$.

☆(d) $\dfrac{1}{x}$ The simplest answer is $\boxed{\ln x}$.

97. Give an example of a function whose derivative is $7x^6 + 8x^3 + 9$. $\boxed{x^7 + 2x^2 + 9x}$ Adding any constant to this also gives a correct answer. This includes $x^7 + 2x^2 + 9x + 1$ and $x^7 + 2x^2 + 9x + \sqrt{37}$ and $x^7 + 2x^2 + 9x - 58$, etc.

98. Is $x^3 - x^{1/3}$ continuous everywhere? Yes Is it differentiable everywhere? No because $\frac{dy}{dx}$ does not exist at $x = 0$.

99. If $f(x) = 8x^4 - x^2$, for what values of x does $f(x) = 0$? $\frac{-1}{2\sqrt{2}}, 0, \frac{1}{2\sqrt{2}}$

For what values of x does $f'(x) = 0$? $\frac{-1}{4}, 0, \frac{1}{4}$

100. For the function $f(x) = x^3$ and $g(x) = 2x^2$, ...

(a) Calculate the derivative of f. $3x^2$

(b) Calculate the derivative of g. $4x$

(c) Calculate the derivative of

$$f(x) + g(x) = x^3 + 2x^2.$$

$3x^2 + 4x$

(d) Calculate the derivative of

$$f(x) \cdot g(x) = 2x^5.$$

$10x^4$

(e) Does $(f + g)' = f' + g'$? In other words, is your answer to (c) the same as adding your answers to (a) and (b)? **Yes**

(f) Does the derivative of a sum equal the sum of the derivatives? **Yes**

(g) Does $(f \cdot g)' = f' \cdot g'$? In other words, is your answer to (d) the same as multiplying your answers to (a) and (b)? **No!**

(h) Does $\frac{d}{dx}[f \cdot g] = \frac{df}{dx} \cdot \frac{dg}{dx}$? This is exactly the same question as (g). **No!**

(i) Does the derivative of a product equal the product of the derivatives? **No!**