Analysis 1, Summer 2023

List 3

Derivatives

☆73. Use the (ε, δ) definition of a limit to show that the limit of

$$f(x) = 4x - 3$$

as x approaches 2 is equal to 5.
As a reminder, starred ☆ tasks are ones that I (Adam) believe are beyond the level of an introductory calculus class.

74. Use the limit definition of a derivative (below) to show that the derivative of

$$f(x) = \frac{36}{x+1}$$

at $x = 2$ is equal to -4. **This task is *not* starred.**

75. Without graphing, determine which one of the three equations below has a solution with $0 \leq x \leq 3$.

(A) $x^2 = 4^x$ (B) $x^3 = 5^x$ (C) $x^5 = 6^x$

For a function $f(x)$ and a number a, the **derivative of f at a**, written $f'(a)$, is the slope of the tangent line to $y = f(x)$ at the point $(a, f(a))$ and is calculated as

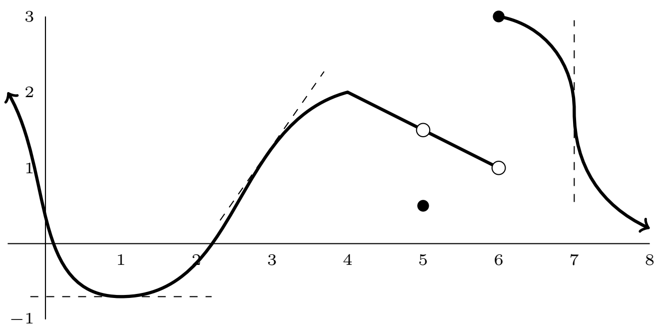
$$f'(a) \quad = \quad \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} \quad = \quad \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

The function $f(x)$ is **differentiable at a** if $f'(a)$ exists and is finite.

76. Calculate $f'(5)$ for the function $f(x) = x^3$.

77. Calculate $f'(1)$ for the function $f(x) = \sqrt{x}$. Hint: See Task 50(b).

78. The graph of a function is shown below. Near $x = 1$, $x = 3$, and $x = 7$, part of the tangent lines to the graph at those points is shown as a dashed line segment.



(a) List all points where the function is not continuous.

(b) List all points where the function is not differentiable (that is, where the derivative does not exist).

79. List all points where $f(x) = \dfrac{|x| - 4}{|x - 4|}$ is not differentiable.

80. (a) If $S(x) = f(x) + g(x)$, does that mean that $S'(3) = f'(3) + g'(3)$? That is, is

$$\lim_{h\to 0} \frac{\big(f(3+h)+g(3+h)\big) - \big(f(3)+g(3)\big)}{h} = \lim_{h\to 0} \frac{f(3+h)-f(3)}{h} + \lim_{h\to 0} \frac{g(3+h)-g(3)}{h}$$

always true?

(b) If $P(x) = f(x) \cdot g(x)$, does that mean that $P'(3) = f'(3) \cdot g'(3)$? That is, is

$$\lim_{h\to 0} \frac{\big(f(3+h)\cdot g(3+h)\big) - \big(f(3)\cdot g(3)\big)}{h} = \lim_{h\to 0} \frac{f(3+h)-f(3)}{h} \cdot \lim_{h\to 0} \frac{g(3+h)-g(3)}{h}$$

always true?

The **linear approximation to $f(x)$ near $x = a$** is the function

$$L(x) = f(a) + f'(a)(x - a).$$

The line $y = L(x)$ is the **tangent line** to $y = f(x)$ at the point $(a, f(a))$.

81. Graph the curve $y = \sqrt{x}$ and the line tangent to that curve at $(1, 1)$.

82. (a) Give the linear approximation to $\sqrt{x}$ near $x = 1$.

(b) Use the approximation from part (a) to estimate $\sqrt{1.2}$.

(c) The true value of $\sqrt{1.2}$ is 1.09545..., so is $L(1.2)$ a good approximation?

(d) Use the approximation from part (a) to estimate $\sqrt{8}$.

(e) The true value of $\sqrt{8}$ is 2.82843..., so is $L(8)$ a good approximation?

83. If f is a function with $f(-4) = 2$ and $f'(-4) = \frac{1}{3}$, give the linear approximation to $f(x)$ near $x = -4$.

84. If g is a function with $g(5) = 12$ and $g'(5) = 2$, use a linear approximation to estimate the value of $g(4.9)$.

85. Give an equation for the tangent line to $y = 4x^2 - x$ at $x = 2$.

86. Give an equation for the tangent line to $7x + 2$ through the point $(30, 212)$.

The Constant Multiple Rule: If c is a constant then

$$(c\,f)' = c\,f' \qquad \big(c\,f(x)\big)' = c\,f'(x) \qquad \tfrac{\mathrm{d}}{\mathrm{d}x}\big[c\,f\big] = c\,\tfrac{\mathrm{d}f}{\mathrm{d}x} \qquad D\big[c\,f\big] = c\,D[f]$$

(these are four ways of writing exact the same fact).

The Sum Rule: $\frac{\mathrm{d}}{\mathrm{d}x}\big[f+g\big] = \frac{\mathrm{d}}{\mathrm{d}x}[f] + \frac{\mathrm{d}}{\mathrm{d}x}[g]$.

The Power Rule: If p is a constant then $\frac{\mathrm{d}}{\mathrm{d}x}\big[x^p\big] = p\,x^{p-1}$.

87. All parts of this task have exactly the same answer!

(a) Find $f'(x)$ for the function $f(x) = 2x^7$.

(b) Give f' if $f = 2x^7$.

(c) Find y' for $y = 2x^7$.

(d) Compute $\frac{\mathrm{d}f}{\mathrm{d}x}$ for the function $f(x) = 2x^7$.

(e) Compute $\frac{\mathrm{d}y}{\mathrm{d}x}$ for $y = 2x^7$.

(f) Give the derivative of $2x^7$ with respect to x.

(g) Find the derivative of $2x^7$.

(h) Calculate $\frac{\mathrm{d}}{\mathrm{d}x} 2x^7$. (i) Calculate $\left(2x^7\right)'$. (j) Calculate $D\left[2x^7\right]$.

(k) Differentiate $2x^7$ with respect to x.

(ℓ) Differentiate $2x^7$.

88. Differentiate $x^5 + \frac{2}{9}x^3 + \sqrt{3x} + \dfrac{x^{10}}{\sqrt{x}}$.

89. Differentiate $\left(x + \sqrt{x}\right)^2$.

☆90. Differentiate $\left(x + \sqrt{x}\right)^{100}$.

91. For each of the functions below, can the Power Rule and/or Constant Multiple Rule (along with maybe some algebra) be used to find the derivative? If so, give the derivative.

(a) $2x^6$ (e) $x^{\sin x}$ (i) $\sin(5\cos(x))$ (m) $\ln(2 + x)$

(b) $2\sqrt{x}$ (f) $(\sin x)^x$ (j) $e^{5\ln(x)}$ (n) $\ln(2x)$

(c) $\sqrt{5x}$ (g) e^x (k) $\dfrac{3}{x^6}$ (o) $\ln(2^x)$

(d) x^π (h) $\cos(5x)$ (ℓ) x^x (p) $\ln(x^2)$

92. Is it possible to find the derivative of the following functions using the Power Rule, Constant Multiple Rule, and Sum Rule?

(a) $x + \ln(5e^x)$ (b) $\dfrac{2x}{x+6}$ (c) $\dfrac{x+6}{2x}$ (d) $\dfrac{x + \frac{1}{x}}{\sqrt{x}}$

93. Give an equation for the tangent line to $y = x^3 - x$ at $x = 2$.

☆94. Find a line that is tangent to both $y = x^2 + 20$ and $y = x^3$.

95. Give the derivative of each of the following functions.

(a) x^{7215}

(b) $5x^{100} + 9x$

(c) $2x^3 - 6x^2 + 10x + 1$

(d) $3\sqrt{x}$

(e) $\sqrt[3]{x}$

(f) $\sqrt{x}^3$

(g) 31

(h) $x + \frac{1}{x}$

(i) $\sqrt{x} + \frac{1}{\sqrt{x}}$

(j) $(3x + 7)^2$

96. Give an example of a function whose derivative is...

(a) x^2 (b) $\sqrt{x}$ (c) $\dfrac{1}{x^2}$ ☆(d) $\dfrac{1}{x}$

97. Give an example of a function whose derivative is $7x^6 + 8x^3 + 9$.

98. Is $x^3 - x^{1/3}$ continuous everywhere? Is it differentiable everywhere?

99. If $f(x) = 8x^4 - x^2$, for what values of x does $f(x) = 0$?
For what values of x does $f'(x) = 0$?

100. For the function $f(x) = x^3$ and $g(x) = 2x^2$, ...

(a) Calculate the derivative of f.

(b) Calculate the derivative of g.

(c) Calculate the derivative of

$$f(x) + g(x) = x^3 + 2x^2.$$

(d) Calculate the derivative of

$$f(x) \cdot g(x) = 2x^5.$$

(e) Does $(f + g)' = f' + g'$? In other words, is your answer to (c) the same as adding your answers to (a) and (b)?

(f) Does the derivative of a sum equal the sum of the derivatives?

(g) Does $(f \cdot g)' = f' \cdot g'$? In other words, is your answer to (d) the same as multiplying your answers to (a) and (b)?

(h) Does $\frac{\mathrm{d}}{\mathrm{d}x}\left[f \cdot g\right] = \frac{\mathrm{d}f}{\mathrm{d}x} \cdot \frac{\mathrm{d}g}{\mathrm{d}x}$?

(i) Does the derivative of a product equal the product of the derivatives?