Analysis 1, Summer 2023

List 7

L'Hôpital's Rule, Taylor series and polynomials

168. Give an equation for the tangent line to $y = e^{3x}(\cos(4x))^5$ at $x = 1$.

169. Is $y = e^{\sin(x)}$ concave up or concave down when $x = \pi$?

170. Find the absolute extremes of $x\ln(x)$ on...
   (a) the interval $[0, \frac{1}{2}]$.
   (b) the interval $[0, 1]$.
   (c) the interval $[0, 2]$.
   (d) the interval $[1, 2]$.

171. Find the inflection points of $f(x) = \frac{3}{10}x^5 - 5x^4 + 32x^3 - 96x^2 + 28$.

172. If f is a smooth function with

x	-2	-1	0	1	2	3	4
f	3	5	-3	7	8	9	12
f'	2	0	-1	-1	1	3	0
f''	0	4	1	-1	$\frac{-8}{3}$	0	1

answer the following:
   (a) Does f have a critical point at $x = 0$?
   (b) Does f have a local minimum at $x = -1$?
   (c) Does f have a local maximum at $x = 4$?
   (d) It it possible that f has an absolute minimum at $x = -1$?
   (e) It it possible that f has an absolute maximum at $x = -1$?
   (f) It it possible that f has an inflection point at $x = 3$?
   (g) It it possible that f has an inflection point at $x = 4$?

L'Hôpital's Rule: if $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$ and $\lim_{x\to a} \frac{f(x)}{g(x)}$ exists, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

The same substitution works if $\lim_{x\to a} f(x) = \infty$ or $-\infty$ and $\lim_{x\to a} g(x) = \infty$ or $-\infty$. And also for one-sided limits and for $x \to \infty$ and $x \to -\infty$.

173. Calculate $\lim_{x\to 1} \dfrac{3x^3 + 4x^2 - 13x + 6}{2x^4 + x^3 - x^2 + x - 3}$ and $\lim_{x\to 4} \dfrac{\sin(\pi x)}{\ln(x-3)}$.

174. Calculate the following limits:

(a) $\lim_{x\to 0^+} \dfrac{\ln(x)}{1/x}$ (b) $\lim_{x\to 0^+} x\ln(x)$ (c) $\lim_{x\to 0^+} e^{x\ln(x)}$ (d) $\lim_{x\to 0^+} x^x$

Hint for (c): recall that $\lim_{x\to a} f(g(x)) = f\big(\lim_{x\to a} g(x)\big)$ if f is continuous.

175. (a) Find $\lim_{x\to 1} \dfrac{x^2-18}{3x+4}$. (b) Find $\lim_{x\to 1} \dfrac{2x}{3}$.

(b) Why are the answers to (a) and (b) not equal?

176. Find $\lim_{x\to 0} \dfrac{2\sin(x)-\sin(2x)}{x-\sin(x)}$.

177. (a) Calculate $\lim_{n\to\infty} n\cdot\ln\big(1+\frac{1}{n}\big)$ using L'Hôpital.

(b) Calculate $\lim_{n\to\infty} \left(1+\dfrac{1}{n}\right)^n$ using the fact that $f(n) = e^{\ln(f(n))}$ and therefore

$$\lim_{n\to\infty} f(n) \;=\; \lim_{n\to\infty} e^{\ln(f(n))} \;=\; e^{\left(\lim_{n\to\infty} \ln(f(n))\right)}.$$

178. For the function $f(x) = x^2e^{-x}$, find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to 0} f(x)$ and $\lim_{x\to -\infty} f(x)$.

For a function $f(x)$, the **degree-N Taylor polynomial around $x = a$** is

$$\sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n,$$

where $n! = n\cdot(n-1)\cdots 2\cdot 1$ is a factorial and $f^{(n)}$ is the n^{th} derivative of f. Note that $0! = 1$ and that $f^{(0)} = f$. In expanded form, this is

$$f(a) + f'(a)\,(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(N)}(a)}{N!}(x-a)^N.$$

179. (a) Calculate the functions $f'(x)$ and $f''(x)$ for $f(x) = x^{5/2}$.

(b) Calculate the numbers $f(4)$, $f'(4)$, and $f''(4)$ for $f(x) = x^{5/2}$.

(c) Give the degree-2 Taylor polynomial for $x^{5/2}$ around $x = 4$. (You may leave "$(x-4)$" in your answer; you do not have to expand it to "$\_\_x^2 + ...$".)

180. Give the degree-3 Taylor polynomial for $e^x\cos(x)$ around $x = 0$. (You will first need to find $f'(x)$, $f''(x)$, $f'''(x)$ and the numbers $f(0)$, ..., $f'''(0)$.)

181. (a) Give the quadratic Taylor polynomial for $\sqrt{x}$ around $x = 1$.

(b) Plug $x = 1.2$ into your polynomial from part (a) to get a "quadratic approximation" to $\sqrt{1.2}$.

(c) Compare the quadratic approximation to the linear approximation from Task 82(a)-(c). Which is closer to the true value of $\sqrt{1.2} \approx 1.09545$?

The **Taylor series around $x = a$** is $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$. Here are Taylor[1] series around zero for some common functions:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots \qquad \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

182. Give the Taylor series for $\frac{x^3}{1-x}$ around $x = 0$.

183. Give the Taylor series for $\ln(1 + x^2)$ around $x = 0$.

184. Give the Taylor polynomial of degree 6 for $f(x) = \ln(x)$ around $x = 1$.

185. (a) Give the Taylor polynomial of degree 3 for $f(x) = \frac{x}{\cos(x)}$ around $x = 0$.

     (b) Give the Taylor polynomial of degree 4 for $f(x) = \frac{\sin(x)}{x}$ around $x = 0$.

     (c) Which more difficult—part (a) or part (b)?

186. On a single set of axes with $x \in [0, 4]$ and $y \in [-1, 2]$, draw the curve $y = \ln(x)$, the tangent line to $y = \ln(x)$ at the point $(2, \ln 2)$, and the graph of the quadratic Taylor polynomial for $\ln(x)$ around $x = 2$.

An **anti-derivative** of $f(x)$ is a function whose derivative is $f(x)$.
In symbols, $F(x)$ is an anti-derivative of $f(x)$ if $F'(x) = f(x)$.

187. (a) Give an anti-derivative of $10x^9$.
     That is, give a function $F(x)$ for which $F'(x) = 10x^9$.

     (b) Give another anti-derivative of $10x^9$.

     (c) Give another anti-derivative of $10x^9$.

     (d) Give another anti-derivative of $10x^9$.

188. Give an anti-derivative of $\sin(x)$.

189. Give an anti-derivative for each of the following functions:

(a) x^3
(b) $12x^5$
(c) $12x^4$
(d) x^{15}
(e) $-3x^{15}$
(f) $\frac{1}{2}x^2$
(g) x^{5000}
(h) $\frac{3}{5}x^{12}$
(i) $\frac{-4}{3}x^7$
(j) $5\sin(x)$
(k) $2\cos(x)$
(ℓ) e^x

190. Give an anti-derivative of $3x^2\cos(x^3 + 9)$. Hint: Think about the Chain Rule.

[1] A Taylor series around zero is also called a "Maclaurin series".