Analysis 1, Summer 2023

List 8

Definite and indefinite integrals, substitution

An **indefinite integral** describes all the anti-derivatives of a function. We write

$$\int f(x)\,dx = F(x) + C,$$

where $F(x)$ is any function for which $F'(x) = f(x)$.

191. Find $\displaystyle\int (2x^5 + 3x - 9)\,\mathrm{d}x = \boxed{\frac{1}{3}x^6 + \frac{3}{2}x^2 - 9x + C}$

192. Find $\displaystyle\int (2u^5 + 3u - 9)\,\mathrm{d}u = \boxed{\frac{1}{3}u^6 + \frac{3}{2}u^2 - 9u + C}$

193. Give each of the following indefinite integrals using basic derivative knowledge:

(a) $\displaystyle\int x^{372.5}\,\mathrm{d}x = \boxed{\frac{1}{373.5}x^{373.5} + C}$

(b) $\displaystyle\int \frac{1}{x}\,\mathrm{d}x = \boxed{\ln(x) + C}$

(c) $\displaystyle\int e^x\,\mathrm{d}x = \boxed{e^x + C}$

(d) $\displaystyle\int 97^x\,\mathrm{d}x = \boxed{\frac{1}{\ln(97)}97^x + C}$

(e) $\displaystyle\int -\sin(x)\,\mathrm{d}x = \boxed{\cos(x) + C}$

(f) $\displaystyle\int \sin(x)\,\mathrm{d}x = \boxed{-\cos(x) + C}$

(g) $\displaystyle\int \cos(x)\,\mathrm{d}x = \boxed{\sin(x) + C}$

(h) $\displaystyle\int 5t^9\,\mathrm{d}t = \boxed{\tfrac{1}{2}t^{10} + C}$

194. If $u = 6x^2 - 5$, give a formula for $\mathrm{d}u$ (this formula will have x and $\mathrm{d}x$ in it) and a formula for $\mathrm{d}x$ (this formula will have x and $\mathrm{d}u$ in it).

$\boxed{\mathrm{d}u = 12x\,\mathrm{d}x}$ and $\boxed{\mathrm{d}x = \dfrac{\mathrm{d}u}{12x}}$

The notation $\boldsymbol{g(x)}\Big|_{\boldsymbol{x=a}}^{\boldsymbol{x=b}}$ or $g(x)\Big|_a^b$ means to do the subtraction $g(b) - g(a)$.

195. Calculate $\frac{1}{3}x^3\Big|_{x=1}^{x=3} = \frac{1}{3}(3)^3 - \frac{1}{3}(1)^3 = 9 - \frac{1}{3} = \boxed{\dfrac{26}{3}}$

196. Calculate $(x^3 + \frac{1}{2}x)\Big|_{x=1}^{x=5}$. $\boxed{126}$

197. Calculate $\left.\dfrac{1-x}{e^x}\right|_{x=0}^{x=1}$. $\boxed{-1}$

The **definite integral** $\displaystyle\int_a^b f(x)\,dx$, spoken as "the integral from a to b of $f(x)$ with respect to x", is the (signed) area of the region with $x = a$ on the left, $x = b$ on the right, $y = 0$ at the bottom, and $y = f(x)$ at the top (but if $f(x) < 0$ for some x or if $b < a$ then it's possible for the area to be negative).
The **Fundamental Theorem of Calculus** says that

$$\int_a^b f(x)\,\mathrm{d}x \;=\; F(x)\Big|_{x=a}^{x=b} \;=\; F(b) - F(a),$$

where $F(x)$ is any function for which $F'(x) = f(x)$.

198. Calculate $\displaystyle\int_1^3 x^2\,\mathrm{d}x$ using the FTC. This is exactly **Task 195**. Answer: $\boxed{\dfrac{26}{3}}$

199. Write, in symbols, the integral from zero to six of x^2 with respect to x, then find the value of that definite integral. $\boxed{\displaystyle\int_0^6 x^2\,\mathrm{d}x = 72}$

200. Evaluate (meaning find of the value of) the following definite integrals using common area formulas.

(a) $\displaystyle\int_3^9 2\,\mathrm{d}x = \boxed{12}$

(b) $\displaystyle\int_3^9 -2\,\mathrm{d}x = \boxed{-12}$

(c) $\displaystyle\int_0^5 x\,\mathrm{d}x = \boxed{\dfrac{25}{2}}$

(d) $\displaystyle\int_{-2}^4 |x|\,\mathrm{d}x = \boxed{10}$

(e) $\displaystyle\int_{-2}^4 x\,\mathrm{d}x = \boxed{6}$

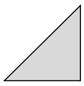
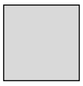
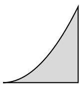
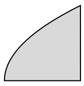
(f) $\displaystyle\int_0^5 3x\,\mathrm{d}x = \boxed{\dfrac{75}{2}}$

(g) $\displaystyle\int_1^5 3x\,\mathrm{d}x = \boxed{36}$

(h) $\displaystyle\int_{-4}^4 \sqrt{16-x^2}\,\mathrm{d}x = \boxed{8\pi}$

(i) $\displaystyle\int_0^7 \sqrt{49-x^2}\,\mathrm{d}x = \boxed{\dfrac{49}{4}\pi}$

201. Match the shapes (a)-(d) with the integral (I)-(IV) that is most likely to calculate its area.

(a)  (b)  (c)  (d)  $\boxed{\text{a-II, b-IV, c-III, d-I}}$

(I) $\int_0^1 \sqrt{x}\,\mathrm{d}x$ (II) $\int_0^1 x\,\mathrm{d}x$ (III) $\int_0^1 x^2\,\mathrm{d}x$ (IV) $\int_0^1 1\,\mathrm{d}x$

202. Find $\int_0^1 \sqrt{x}\,\mathrm{d}x$. $\boxed{\frac{2}{3}}$

203. Evaluate the following definite integrals using the FTC. Your answer for each should be a number.

(a) $\int_{-3}^{9} 2\,\mathrm{d}x = 2x\Big|_{x=-3}^{x=9} = 18 - (-6) = \boxed{24}$

(b) $\int_1^5 3x\,\mathrm{d}x = \frac{3}{2}x^2\Big|_{x=1}^{x=5} = \frac{75}{2} - \frac{3}{2} = \boxed{36}$

(c) $\int_1^{12} \frac{1}{x}\,\mathrm{d}x = \ln(x)\Big|_{x=1}^{x=12} = \ln(12) - \ln(0) = \boxed{\ln(12)}$

(d) $\int_0^9 (x^3 - 9x)\,\mathrm{d}x = (\frac{1}{4}x^4 - \frac{9}{2}x^2)\Big|_{x=0}^{x=9} = \frac{5103}{4} - 0 = \boxed{\frac{5103}{4}}$

(e) $\int_0^{\pi} \sin(t)\,\mathrm{d}t = (-\cos(t))\Big|_{t=0}^{t=\pi} = -\cos(\pi) - (-\cos(0)) = \boxed{2}$

(f) $\int_2^8 3\cdot\sqrt{u}\,\mathrm{d}u = 2x^{3/2}\Big|_{x=2}^{x=8} = 32\sqrt{2} - 4\sqrt{2} = \boxed{28\sqrt{2}}$

(g) $\int_0^1 (e^x + x^e)\,\mathrm{d}x = \boxed{\frac{e^2}{1+e}}$

(h) $\int_{-1}^{1} x^2\,\mathrm{d}x = \boxed{\frac{2}{3}}$

(i) $\int_1^3 t\,\mathrm{d}t = \boxed{4}$

(j) $\int_9^9 \sin(x^2)\,\mathrm{d}x = \boxed{0}$

(k) $\int_0^5 \cos(x)\,\mathrm{d}x = \boxed{\sin(5)}$

204. Evaluate the following definite integrals using the FTC. Your answers will be formulas.

(a) $\int_a^9 2\,\mathrm{d}x = 2x\Big|_{x=a}^{x=9} = \boxed{18 - 2a}$ (b) $\int_1^5 kx\,\mathrm{d}x = \frac{k}{2}x^2\Big|_{x=1}^{x=5} = \frac{25k}{2} - \frac{k}{2} = \boxed{12k}$

(c) $\int_1^t \frac{1}{x}\,\mathrm{d}x = \ln(x)\Big|_{x=1}^{x=t} = \ln(t) - \ln(1) = \boxed{\ln(t)}$

(d) $\int_0^9 (x^p - qx)\,\mathrm{d}x = \boxed{\frac{9^{p+1}}{p+1} - \frac{81q}{2}}$

205. If $\int_1^4 f(x)\,\mathrm{d}x = 12$ and $\int_1^6 f(x)\,\mathrm{d}x = 15$, what is the value of $I = \int_4^6 f(x)\,\mathrm{d}x$?

$\int_1^6 f(x)\,\mathrm{d}x = \int_1^4 f(x)\,\mathrm{d}x + \int_4^6 f(x)\,\mathrm{d}x$, so $12 + I = 14$, and thus $I = \boxed{3}$.

206. If $\int_0^1 f(x)\,\mathrm{d}x = 7$ and $\int_0^1 g(x)\,\mathrm{d}x = 3$, calculate each of the following or say that there is not enough information to possibly do the calculation.

(a) $\int_0^1 \big(f(x) + g(x)\big)\,\mathrm{d}x = \boxed{10}$

(b) $\int_0^1 \big(f(x) - g(x)\big)\,\mathrm{d}x = \boxed{4}$

(c) $\int_0^1 \big(f(x) \cdot g(x)\big)\,\mathrm{d}x$ $\boxed{\text{not enough info}}$

(d) $\int_0^1 \big(5f(x)\big)\,\mathrm{d}x = \boxed{35}$

(e) $\int_0^1 \big(f(x)^5\big)\,\mathrm{d}x$ $\boxed{\text{not enough info}}$

207. Simplify $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\int_1^t \dfrac{1}{x}\,\mathrm{d}x\right)$ to a formula that does not include x (assume $t > 1$). $\boxed{\dfrac{1}{t}}$

208. Simplify $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\int_3^t \dfrac{\sin(x)}{x}\,\mathrm{d}x\right)$ to a formula that does not include x (assume t $\boxed{\dfrac{\sin t}{t}}$

209. Simplify $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^{t^2} \sin(x)\,\mathrm{d}x\right)$ and $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^{t^2} \sin(x^2)\,\mathrm{d}x\right)$ to formulas that do not include x. $\boxed{2t\sin(t^2)}$ and $\boxed{2t\sin(t^4)}$

210. Given that $\int \ln(x)\,\mathrm{d}x = x\ln(x) - x + C$, evaluate $\int_1^{e^5} \ln(x)\,\mathrm{d}x$.

$x\ln(x) - x\Big|_{x=1}^{x=e^5} = e^5\ln(e^5) - e^5 - (\ln(1) - 1) = 5e^5 - e^5 - (-1) = \boxed{4e^5 + 1}$

Substitution: $\int f(u(x)) \cdot u'(x)\,\mathrm{d}x = \int f(u)\,\mathrm{d}u$

211. (a) Re-write $\int \dfrac{x}{(6x^2-5)^3}\,\mathrm{d}x$ as $\int \ldots \mathrm{d}u$ using the substitution $u = 6x^2 - 5$.

$\mathrm{d}u = 12x\,\mathrm{d}x$, so $x\,\mathrm{d}x = \frac{1}{12}\,\mathrm{d}u$ and the integral is $\boxed{\int \dfrac{1}{12u^3}\,\mathrm{d}u}$.

(b) Find $\int \dfrac{x}{(6x^2-5)^3}\,\mathrm{d}x$. (Your final answer should not have u at all.) $\boxed{\dfrac{-1}{24(6x^2-5)^2}}$

212. (a) Re-write $\int x^3 \sin(x^4)\,\mathrm{d}x$ as $\int \ldots \mathrm{d}u$ using the substitution $u = x^4$. $\boxed{\int \frac{1}{4}\sin(u)\,\mathrm{d}u}$

(b) Find $\int x^3 \sin(x^4)\,\mathrm{d}x = \boxed{-\frac{1}{4}\cos(x^4) + C}$

213. (a) Re-write $\int x \sin(x^4)\,\mathrm{d}x$ as $\int \ldots \mathrm{d}u$ using the substitution $u = x^2$. $\boxed{\int \frac{1}{2}\sin(u^2)\,\mathrm{d}u}$

☆(b) Find $\int x \sin(x^4)\,\mathrm{d}x$ There is literally no "elementary" formula for this. You might see $\sqrt{\pi/8}S(x^2\sqrt{2/\pi})$ in some sources, but this is just re-writing the integral using a special short-hand for this "Fresnel" integral.

214. Find $\int \frac{x^4 - x^3 - 1}{4x^5 - 5x^4 - 20x + 3}\,\mathrm{d}x$ using substitution. $= \boxed{\frac{1}{20}\ln(4x^5 - 5x^4 - 20x + 3) + C}$

215. Find $\int \cot(x)\,\mathrm{d}x = \int \frac{\cos(x)}{\sin(x)}\,\mathrm{d}x$ using substitution.

With $u = \sin(x)$, $\mathrm{d}u = \cos(x)\,\mathrm{d}x$, so

$$\int \frac{\cos(x)\,\mathrm{d}x}{\sin(x)} = \int \frac{\mathrm{d}u}{u} = \ln|u| + C = \boxed{\ln|\sin(x)| + C}$$

Technically the "C" could actually be a piecewise function that is constant on each interval where $\ln|\sin(x)|$ is continuous. But it is common to just write "$+C$" anyway.

216. Which of the following has the same value as $\int_2^4 \frac{3x^2 - 2}{\ln(x^3 - 2x + 1)}\,\mathrm{d}x$?

(A) $\int_5^{57} \frac{1}{\ln(u)}\,\mathrm{d}u$ (B) $\int_2^4 \frac{1}{\ln(u)}\,\mathrm{d}u$ (C) $\int_{10}^{46} \frac{1}{\ln(u)}\,\mathrm{d}u$ (D) $\int_1^2 \frac{1}{\ln(u)}\,\mathrm{d}u$

Using the substitution $u = x^3 - 2x + 1$, we have $\mathrm{d}u = (3x^2 - 2)\,\mathrm{d}x$, so

$$\frac{(3x^2 - 2)\,\mathrm{d}x}{\ln(x^3 - 2x + 1)} = \frac{\mathrm{d}u}{\ln(u)}.$$

The values 2 and 4 are x-values, so **these change** when we write $\int \ldots \mathrm{d}u$.

When $x = 2$, $u = 2^3 - 2(2) + 1 = 5$. When $x = 4$, $u = 4^3 - 2(4) + 1 = 57$. $\boxed{\text{(A)}}$

217. Find the following integrals using substitution:

(a) $\int (5 - x)^{10}\,\mathrm{d}x = \int -u^{10}\,\mathrm{d}u = \frac{-1}{11}u^{11} + C = \boxed{\frac{-1}{11}(5 - x)^{11} + C}$

using $u = 5 - x$, so $\mathrm{d}u = -\,\mathrm{d}x$.

(b) $\int_1^3 \frac{x}{(6x^2 - 5)^3}\,\mathrm{d}x$

Indefinite: $\int \frac{x}{(6x^2 - 5)^3}\,\mathrm{d}x = \int \frac{1}{12}u^{-3}\,\mathrm{d}u = \frac{-1}{24}u^{-2} + C = \frac{-1}{24(6x^2 - 5)^2} + C$

using $u = 6x^2 - 5$, so $\mathrm{d}u = 12x\,\mathrm{d}x$. Then the definite integral is

$$\int_1^3 \frac{x}{(6x^2 - 5)^3}\,\mathrm{d}x = \frac{-1}{24(6x^2 - 5)^2}\Bigg|_{x=1}^{x=3} = \boxed{\frac{100}{2401}}.$$

Alternatively, when $x = 1, u = 6(1)^2 - 5 = 1$ and when $x = 3, u = 6(3)^2 - 5 = 49$, so this is

$$\int_1^{49} \tfrac{1}{12}u^{-3}\,\mathrm{d}u = \frac{-1}{24u^2}\bigg|_{u=1}^{u=49} = \boxed{\frac{100}{2401}}.$$

(c) $\displaystyle\int \sqrt{4x+3}\,\mathrm{d}x = \int \tfrac{1}{4}u^{1/2}\,\mathrm{d}u = \tfrac{1}{6}u^{3/2} + C = \boxed{\tfrac{1}{6}(4x+3)^{3/2} + C}$

using $u = 4x + 3$, so $\mathrm{d}u = 4\,\mathrm{d}x$.

(d) $\displaystyle\int_0^{\sqrt{\pi}} x\sin(x^2)\,\mathrm{d}x$

Indefinite: $\displaystyle\int x\sin(x^2)\,\mathrm{d}x = \int \tfrac{1}{2}\sin(u)\,\mathrm{d}u = \tfrac{-1}{2}\cos(u) + C = \tfrac{-1}{2}\cos(x^2) + C$

using $u = x^2$, so $\mathrm{d}u = 2x\,\mathrm{d}x$. Then the definite integral is

$$\int_0^{\sqrt{\pi}} x\sin(x^2)\,\mathrm{d}x = \tfrac{-1}{2}\cos(x^2)\bigg|_{x=0}^{x=\sqrt{\pi}} = \tfrac{-1}{2}(-1) - \tfrac{-1}{2}(1) = \boxed{1}.$$

Alternatively, when $x = 0, u = 0$ and when $x = \sqrt{\pi}, u = \pi$, so this is

$$\int_0^{\pi} \tfrac{1}{2}\sin(u)\,\mathrm{d}u = \tfrac{-1}{2}\cos(u)\bigg|_{u=0}^{u=\pi} = \tfrac{-1}{2}(-1) - \tfrac{-1}{2}(1) = \boxed{1}.$$

(e) $\displaystyle\int \frac{5}{4x+9}\,\mathrm{d}x = \boxed{\frac{5}{4}\ln(4x+9) + C}$

(f) $\displaystyle\int \frac{5x}{4x^2+9}\,\mathrm{d}x = \boxed{\frac{5}{8}\ln(4x^2+9) + C}$

☆ (g) $\displaystyle\int \frac{5}{4x^2+9}\,\mathrm{d}x = \boxed{\frac{5}{6}\arctan(\tfrac{2}{3}x) + C}$

(h) $\displaystyle\int \frac{\sin(\ln(x))}{x}\,\mathrm{d}x = \int \sin(u)\,\mathrm{d}u = -\cos(u) + C = \boxed{-\cos(\ln(x)) + C}$

using $u = \ln(x)$, so $\mathrm{d}u = \frac{1}{x}\,\mathrm{d}x$.

☆ (i) $\displaystyle\int_0^9 \sqrt{4 - \sqrt{x}}\,\mathrm{d}x$

$u = 4 - \sqrt{x}$ gives $\mathrm{d}u = \frac{-1}{2\sqrt{x}}\,\mathrm{d}x$. There is no "$\frac{-1}{2\sqrt{x}}$" in the original integral, but $\mathrm{d}x = -2\sqrt{x}\,\mathrm{d}u$ and $\sqrt{x} = 4 - u$, so $\mathrm{d}x = -2(4-u)\,\mathrm{d}u = (2u-8)\,\mathrm{d}u$. Thus

$$\int \sqrt{4-\sqrt{x}}\,\mathrm{d}x = \int \sqrt{u}\,(2u-8)\mathrm{d}u = \int (2u^{3/2} - 8u^{1/2})\,\mathrm{d}u$$
$$= \tfrac{4}{5}u^{5/2} - \tfrac{16}{3}u^{3/2} + C.$$

The definite integral is

$$\left(\tfrac{4}{5}(4 - x^{1/2})^{5/2} - \tfrac{16}{3}(4 - x^{1/2})^{3/2}\right)\bigg|_{x=0}^{x=9} = \left(\frac{-68}{15}\right) - \left(\frac{-125}{15}\right) = \boxed{\frac{188}{15}}$$

or, with $4 - \sqrt{0} = 4$ and $4 - \sqrt{9} = 1$,

$$\left(\tfrac{4}{5}u^{5/2} - \tfrac{16}{3}u^{3/2}\right)\bigg|_{u=4}^{u-1} = \left(\frac{-68}{15}\right) - \left(\frac{-125}{15}\right) = \boxed{\frac{188}{15}}$$

(j) $\int x^3 \cos(2x^4)\,\mathrm{d}x = \int \frac{1}{8}\cos(u)\,\mathrm{d}u = \frac{1}{8}\sin(u) + C = \boxed{\frac{1}{8}\sin(2x^4) + C}$

using $u = 2x^4$, so $\mathrm{d}u = 8x^3\,\mathrm{d}x$.

(k) $\int e^{t^5} t^4\,\mathrm{d}t = \int e^u \frac{1}{5}\,\mathrm{d}u = \frac{1}{5}e^u + C = \boxed{\frac{1}{5}e^{t^5} + C}$

using $u = t^5$, so $\mathrm{d}u = 5t^4\,\mathrm{d}t$.

(l) $\int \frac{(\ln(x))^2}{5x}\,\mathrm{d}x = \boxed{\frac{(\ln(x))^3}{15}}$

(m) $\int \frac{1}{x\ln(x)}\,\mathrm{d}x = \int \frac{1}{u}\,\mathrm{d}u = \ln(u) + C = \boxed{\ln(\ln(x)) + C}$

using $u = \ln(x)$, so $\mathrm{d}u = \frac{1}{x}\,\mathrm{d}x$.

(n) $\int_0^{\pi/2} \sin(x)\cos(x)\,\mathrm{d}x = \boxed{\frac{1}{2}}$

(o) $\int \sin(1-x)(2-\cos(1-x))^4\,\mathrm{d}x = \boxed{\frac{-1}{5}(2-\cos(1-x))^5 + C}$

(p) $\int (1 - \frac{1}{v})\cos(v - \ln(v))\,\mathrm{d}v = \boxed{\sin(v - \ln(v)) + C}$

(q) $\int \frac{t}{\sqrt{1-4t^2}}\,\mathrm{d}t = \boxed{\frac{-1}{4}\sqrt{1-4t^2} + C}$

(r) $\int_0^{\pi/3} \big(3\sin(\frac{1}{2}x) + 5\cos(x)\big)\,\mathrm{d}x = \boxed{6 - \frac{\sqrt{3}}{2}}$

(s) $\int \frac{e^{\tan(x)}}{\cos(x)^2}\,\mathrm{d}x = \boxed{e^{\tan(x)} + C}$

(t) $\int_1^5 \frac{x^2+1}{x^3+3x}\,\mathrm{d}x = \frac{1}{3}\ln(x^3+3x)\Big|_1^5 = \boxed{\frac{1}{3}\ln(35)}$

218. If $\int_9^{16} f(x)\,\mathrm{d}x = 1$, calculate $\int_3^{10} f(x^2)\,x\,\mathrm{d}x = \boxed{\frac{1}{2}}$

219. If $\int_0^1 f(x)\,\mathrm{d}x = 19$, calculate each of the following or say that there is not enough information to possibly do the calculation.

(a) $\int_0^1 f(x^5)\,5x^4\,\mathrm{d}x = \boxed{19}$

(b) $\int_0^1 f(x^5)\,x^4\,\mathrm{d}x = \boxed{\frac{19}{5}}$

(c) $\int_0^1 f(\frac{1}{5}x^5)\,x^4\,\mathrm{d}x$ not enough info

(d) $\int_0^1 \frac{f(\sqrt{x})}{\sqrt{x}}\,\mathrm{d}x = \boxed{38}$

(e) $\int_0^1 f(\sin(x))\cos(x)\,\mathrm{d}x$ not enough info

(f) $\int_0^1 f(\sin(\frac{\pi}{2}x))\cos(\frac{\pi}{2}x)\,\mathrm{d}x = \boxed{\frac{38}{\pi}}$