Analysis 1, Summer 2023

List 8

Definite and indefinite integrals, substitution

An **indefinite integral** describes all the anti-derivatives of a function. We write

$$\int f(x)\,dx = F(x) + C,$$

where $F(x)$ is any function for which $F'(x) = f(x)$.

191. Find $\displaystyle\int (2x^5 + 3x - 9)\,\mathrm{d}x$.

192. Find $\displaystyle\int (2u^5 + 3u - 9)\,\mathrm{d}u$.

193. Give each of the following indefinite integrals using basic derivative knowledge:

(a) $\displaystyle\int x^{372.5}\,\mathrm{d}x$ (c) $\displaystyle\int e^x\,\mathrm{d}x$ (e) $\displaystyle\int -\sin(x)\,\mathrm{d}x$ (g) $\displaystyle\int \cos(x)\,\mathrm{d}x$

(b) $\displaystyle\int \frac{1}{x}\,\mathrm{d}x$ (d) $\displaystyle\int 97^x\,\mathrm{d}x$ (f) $\displaystyle\int \sin(x)\,\mathrm{d}x$ (h) $\displaystyle\int 5t^9\,\mathrm{d}t$

194. If $u = 6x^2 - 5$, give a formula for $\mathrm{d}u$ (this formula will have x and $\mathrm{d}x$ in it) and a formula for $\mathrm{d}x$ (this formula will have x and $\mathrm{d}u$ in it).

The notation $\boldsymbol{g(x)}\Big|_{\boldsymbol{x=a}}^{\boldsymbol{x=b}}$ or $g(x)\Big|_a^b$ means to do the subtraction $g(b) - g(a)$.

195. Calculate $\frac{1}{3}x^3\Big|_{x=1}^{x=3}$.

196. Calculate $(x^3 + \frac{1}{2}x)\Big|_{x=1}^{x=5}$.

197. Calculate $\displaystyle\frac{1-x}{e^x}\Big|_{x=0}^{x=1}$.

The **definite integral** $\displaystyle\int_a^b f(x)\,dx$, spoken as "the integral from a to b of $f(x)$ with respect to x", is the (signed) area of the region with $x = a$ on the left, $x = b$ on the right, $y = 0$ at the bottom, and $y = f(x)$ at the top (but if $f(x) < 0$ for some x or if $b < a$ then it's possible for the area to be negative).
The **Fundamental Theorem of Calculus** says that

$$\int_a^b f(x)\,\mathrm{d}x \;=\; F(x)\Big|_{x=a}^{x=b} \;=\; F(b) - F(a),$$

where $F(x)$ is any function for which $F'(x) = f(x)$.

198. Calculate $\displaystyle\int_1^3 x^2\,\mathrm{d}x$ using the FTC.

199. Write, in symbols, the integral from zero to six of x^2 with respect to x, then find the value of that definite integral.

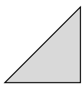
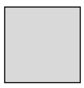
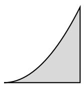
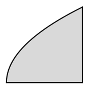
200. Evaluate (meaning find of the value of) the following definite integrals using common area formulas.

(a) $\int_3^9 2\,dx$ (d) $\int_{-2}^4 |x|\,dx$ (g) $\int_1^5 3x\,dx$

(b) $\int_3^9 -2\,dx$ (e) $\int_{-2}^4 x\,dx$ (h) $\int_{-4}^4 \sqrt{16-x^2}\,dx$

(c) $\int_0^5 x\,dx$ (f) $\int_0^5 3x\,dx$ (i) $\int_0^7 \sqrt{49-x^2}\,dx$

201. Match the shapes (a)-(d) with the integral (I)-(IV) that is most likely to calculate its area.

(a)  (b)  (c)  (d) 

(I) $\int_0^1 \sqrt{x}\,dx$ (II) $\int_0^1 x\,dx$ (III) $\int_0^1 x^2\,dx$ (IV) $\int_0^1 1\,dx$

202. Find $\int_0^1 \sqrt{x}\,dx$.

203. Evaluate the following definite integrals using the FTC. Your answer for each should be a number.

(a) $\int_{-3}^9 2\,dx$ (e) $\int_0^\pi \sin(t)\,dt$ (i) $\int_1^3 t\,dt$

(b) $\int_1^5 3x\,dx$ (f) $\int_2^8 3\cdot\sqrt{u}\,du$ (j) $\int_9^9 \sin(x^2)\,dx$

(c) $\int_1^{12} \frac{1}{x}\,dx$ (g) $\int_0^1 (e^x + x^e)\,dx$ (k) $\int_0^5 \cos(x)\,dx$

(d) $\int_0^9 (x^3 - 9x)\,dx$ (h) $\int_{-1}^1 x^2\,dx$

204. Evaluate the following definite integrals using the FTC. Your answers will be formulas.

(a) $\int_a^9 2\,dx$ (b) $\int_1^5 kx\,dx$

(c) $\int_1^t \frac{1}{x}\,dx$ assuming $t > 1$

(d) $\int_0^9 (x^p - qx)\,dx$ assuming $p > -1$

205. If $\int_1^4 f(x)\,dx = 12$ and $\int_1^6 f(x)\,dx = 15$, what is the value of $I = \int_4^6 f(x)\,dx$?

206. If $\int_0^1 f(x)\,\mathrm{d}x = 7$ and $\int_0^1 g(x)\,\mathrm{d}x = 3$, calculate each of the following or say that there is not enough information to possibly do the calculation.

(a) $\int_0^1 \big(f(x)+g(x)\big)\,\mathrm{d}x$ (c) $\int_0^1 \big(f(x)\cdot g(x)\big)\,\mathrm{d}x$ (e) $\int_0^1 \big(f(x)^5\big)\,\mathrm{d}x$

(b) $\int_0^1 \big(f(x)-g(x)\big)\,\mathrm{d}x$ (d) $\int_0^1 \big(5f(x)\big)\,\mathrm{d}x$

207. Simplify $\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_1^t \frac{1}{x}\,\mathrm{d}x\right)$ to a formula that does not include x (assume $t > 1$).

208. Simplify $\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_3^t \frac{\sin(x)}{x}\,\mathrm{d}x\right)$ to a formula that does not include x (assume $t > 3$).

209. Simplify $\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^{t^2} \sin(x)\,\mathrm{d}x\right)$ and $\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^{t^2} \sin(x^2)\,\mathrm{d}x\right)$ to formulas that do not include x.

210. Given that $\int \ln(x)\,\mathrm{d}x = x\ln(x) - x + C$, evaluate $\int_1^{e^5} \ln(x)\,\mathrm{d}x$.

Substitution: $\int f(u(x))\cdot u'(x)\,\mathrm{d}x = \int f(u)\,\mathrm{d}u$

211. (a) Re-write $\int \frac{x}{(6x^2-5)^3}\,\mathrm{d}x$ as $\int \ldots\,\mathrm{d}u$ using the substitution $u = 6x^2 - 5$.

(b) Find $\int \frac{x}{(6x^2-5)^3}\,\mathrm{d}x$. (Your final answer should not have u at all.)

212. (a) Re-write $\int x^3\sin(x^4)\,\mathrm{d}x$ as $\int \ldots\,\mathrm{d}u$ using the substitution $u = x^4$.

(b) Find $\int x^3\sin(x^4)\,\mathrm{d}x$.

213. (a) Re-write $\int x\sin(x^4)\,\mathrm{d}x$ as $\int \ldots\,\mathrm{d}u$ using the substitution $u = x^2$.

☆(b) Find $\int x\sin(x^4)\,\mathrm{d}x$.

214. Find $\int \frac{x^4 - x^3 - 1}{4x^5 - 5x^4 - 20x + 3}\,\mathrm{d}x$ using substitution.

215. Find $\int \cot(x)\,\mathrm{d}x = \int \frac{\cos(x)}{\sin(x)}\,\mathrm{d}x$ using substitution.

216. Which of the following has the same value as $\int_2^4 \frac{3x^2-2}{\ln(x^3-2x+1)}\,\mathrm{d}x$?

(A) $\int_5^{57} \frac{1}{\ln(u)}\,\mathrm{d}u$ (B) $\int_2^4 \frac{1}{\ln(u)}\,\mathrm{d}u$ (C) $\int_{10}^{46} \frac{1}{\ln(u)}\,\mathrm{d}u$ (D) $\int_1^2 \frac{1}{\ln(u)}\,\mathrm{d}u$

217. Find the following integrals using substitution:

(a) $\int (5-x)^{10}\,\mathrm{d}x$

(b) $\int_1^3 \frac{x}{(6x^2-5)^3}\,\mathrm{d}x$

(c) $\int \sqrt{4x+3}\,\mathrm{d}x$

(d) $\int_0^{\sqrt{\pi}} x\sin(x^2)\,\mathrm{d}x$

(e) $\int \frac{5}{4x+9}\,\mathrm{d}x$

(f) $\int \frac{5x}{4x^2+9}\,\mathrm{d}x$

☆(g) $\int \frac{5}{4x^2+9}\,\mathrm{d}x$

(h) $\int \frac{\sin(\ln(x))}{x}\,\mathrm{d}x$

☆(i) $\int_0^9 \sqrt{4-\sqrt{x}}\,\mathrm{d}x$

(j) $\int x^3\cos(2x^4)\,\mathrm{d}x$

(k) $\int e^{t^5}t^4\,\mathrm{d}t$

(l) $\int \frac{(\ln(x))^2}{5x}\,\mathrm{d}x$

(m) $\int \frac{1}{x\ln(x)}\,\mathrm{d}x$

(n) $\int_0^{\pi/2} \sin(x)\cos(x)\,\mathrm{d}x$

(o) $\int \sin(1-x)(2-\cos(1-x))^4\,\mathrm{d}x$

(p) $\int (1-\frac{1}{v})\cos(v-\ln(v))\,\mathrm{d}v$

(q) $\int \frac{t}{\sqrt{1-4t^2}}\,\mathrm{d}t$

(r) $\int_0^{\pi/3} \big(3\sin(\tfrac{1}{2}x)+5\cos(x)\big)\,\mathrm{d}x$

(s) $\int \frac{e^{\tan(x)}}{\cos(x)^2}\,\mathrm{d}x$

(t) $\int_1^5 \frac{x^2+1}{x^3+3x}\,\mathrm{d}x$

218. If $\int_9^{16} f(x)\,\mathrm{d}x = 1$, calculate $\int_3^{10} f(x^2)\,x\,\mathrm{d}x$.

219. If $\int_0^1 f(x)\,\mathrm{d}x = 19$, calculate each of the following or say that there is not enough information to possibly do the calculation.

(a) $\int_0^1 f(x^5)\,5x^4\,\mathrm{d}x$

(b) $\int_0^1 f(x^5)\,x^4\,\mathrm{d}x$

(c) $\int_0^1 f(\tfrac{1}{5}x^5)\,x^4\,\mathrm{d}x$

(d) $\int_0^1 \frac{f(\sqrt{x})}{\sqrt{x}}\,\mathrm{d}x$

(e) $\int_0^1 f(\sin(x))\cos(x)\,\mathrm{d}x$

(f) $\int_0^1 f(\sin(\tfrac{\pi}{2}x))\cos(\tfrac{\pi}{2}x)\,\mathrm{d}x$