Math 1688
Algebra and An. Geometry

Lectures: Thursdays 17:05 - 18:45 with dr Adam Abrams.
Zoom

Exercises K01-67a: Th. 15:15 - 16:55 with dr hab. Oleksii Kulyk
Exercises K01-67b: Th. 18:55 - 20:35 with dr Adam Abrams
both in building D-1 room 311a

Topics

Almost every topic in the class is related in *some* way to one of the following two kinds of problems:

- For what values of x does

$$x^4 - 3x^3 - 12x - 16 = 0\ ?$$

- For what values of (x, y, z) are

$$\begin{aligned} 3x - y + 9z &= 8 \\ 4x + 2y \phantom{{}+ 27z} &= 14 \\ -3x + 6y - 27z &= -3 \end{aligned}$$

  all true?

While discussing these we will find many other useful ideas.

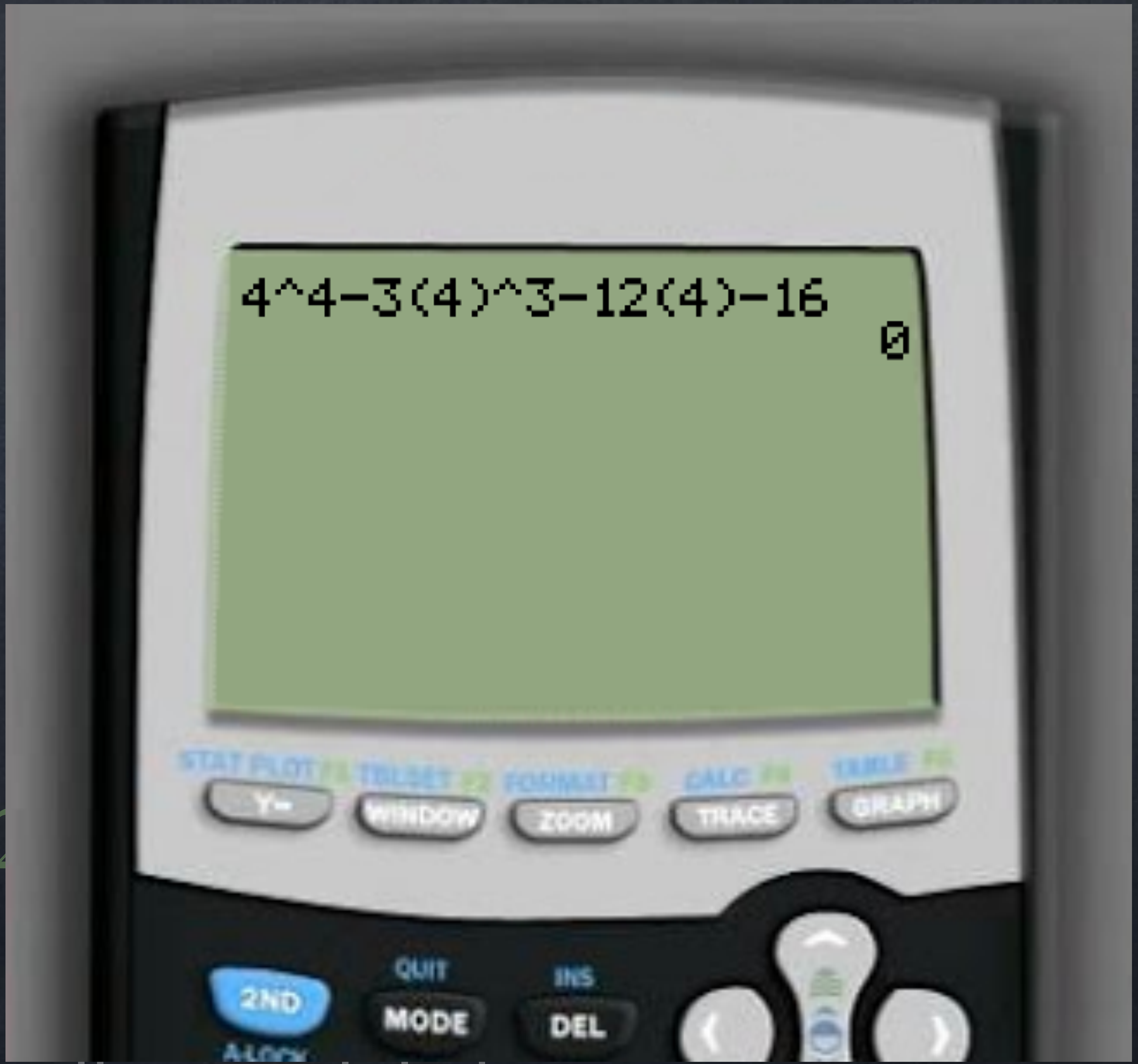
Topics

Almost every topic in the class is related [to the] following two kinds of problems:

- For what values of x does
  $$x^4 - 3x^3 - 12x - 16 = 0?$$

You ***are*** allowed to use calculators in this class, for exercises, quizzes, and exams.

You may ***not*** communicate with other students during quizzes or exams.



While discussing these we will find many other useful ideas.

Topics

Polynomials

- Complex numbers
- Roots zeros of polynomials
- Factoring and remainders

Linear algebra

- Vectors
- Lines and planes
- Systems of linear equations
- Linear maps
- Matrices

Course website

All course policies can be found at

http://theadamabrams.com/teaching/1688

Lecture slides and problem sets will also be posted to this site throughout the semester.

Grading policy

The same grade is used for 1688W and 1688C.

- Eight quizzes (5 points each), but the lowest score is ignored!
- Three exams (15 points, 15, 20).
- Activity points (5 points).

This makes 7×5 + 15+15+20 + 5 = 90 total possible points.

Points	[0, 45)	[45, 54)	[54, 63)	[63, 72)	[72, 81)	[81, 90]
Grade	2.0	3.0	3.5	4.0	4.5	5.0

- More than 4 unexcused absences after this week → grade of 2.0.
- Cheating on exams → grade of 2.0 (cannot be improved).

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Accessibility

Accessibility and Support Department for People with Disabilities

- Office: building C-13 room 109
- Website: https://ddo.pwr.edu.pl/
- Email: pomoc.n@pwr.edu.pl

English language and some polls

poles

Poles

polls

Course overview

From https://wmat.pwr.edu.pl/studenci/kursy-ogolnouczelniane/karty-przedmiotow/studia-stacjonarne

FACULTY OF COMPUTER SCIENCE AND MANAGEMENT

SUBJECT CARD

Name in English	**ALGEBRA AND ANALYTIC GEOMETRY**
Name in Polish	**ALGEBRA Z GEOMETRIĄ ANALITYCZNĄ**
Main field of study (if applicable)	***Computer Science***
Level and form of studies	**I level, full time**

⋮

PROGRAM CONTENT

	Form of classes - lectures	Hours
Lec1	Mathematical induction. Newton's binomial formula.	1
Lec2	The notion of a matrix. Operations on matrices. Transposition. Examples of matrices (triangular, symmetric, diagonal etc.).	2
Lec3	The determinant of a matrix. The Laplace expansion. Cofactor of an element of a matrix. Minors. Properties of determinants. Calculation of determinants by elementary row and column operations. Cauchy's theorem. Nonsingular matrix.	3
Lec4	Inverse matrix. Computation of inverse matrix by cofactors or by elementary row operations. Properties of inverse matrices. Matrix equations. Rank of a matrix. Applications of determinants, their connections with rank and invertibility.	2
Lec5	Systems of linear equations. Rouché–Capelli theorem. Cramer's formulas. Gaussian elimination. Solving arbitrary systems of linear equations.	3
Lec6	Complex numbers. Operations on complex numbers in algebraic form. Complex	2

will not be on quizzes or exams

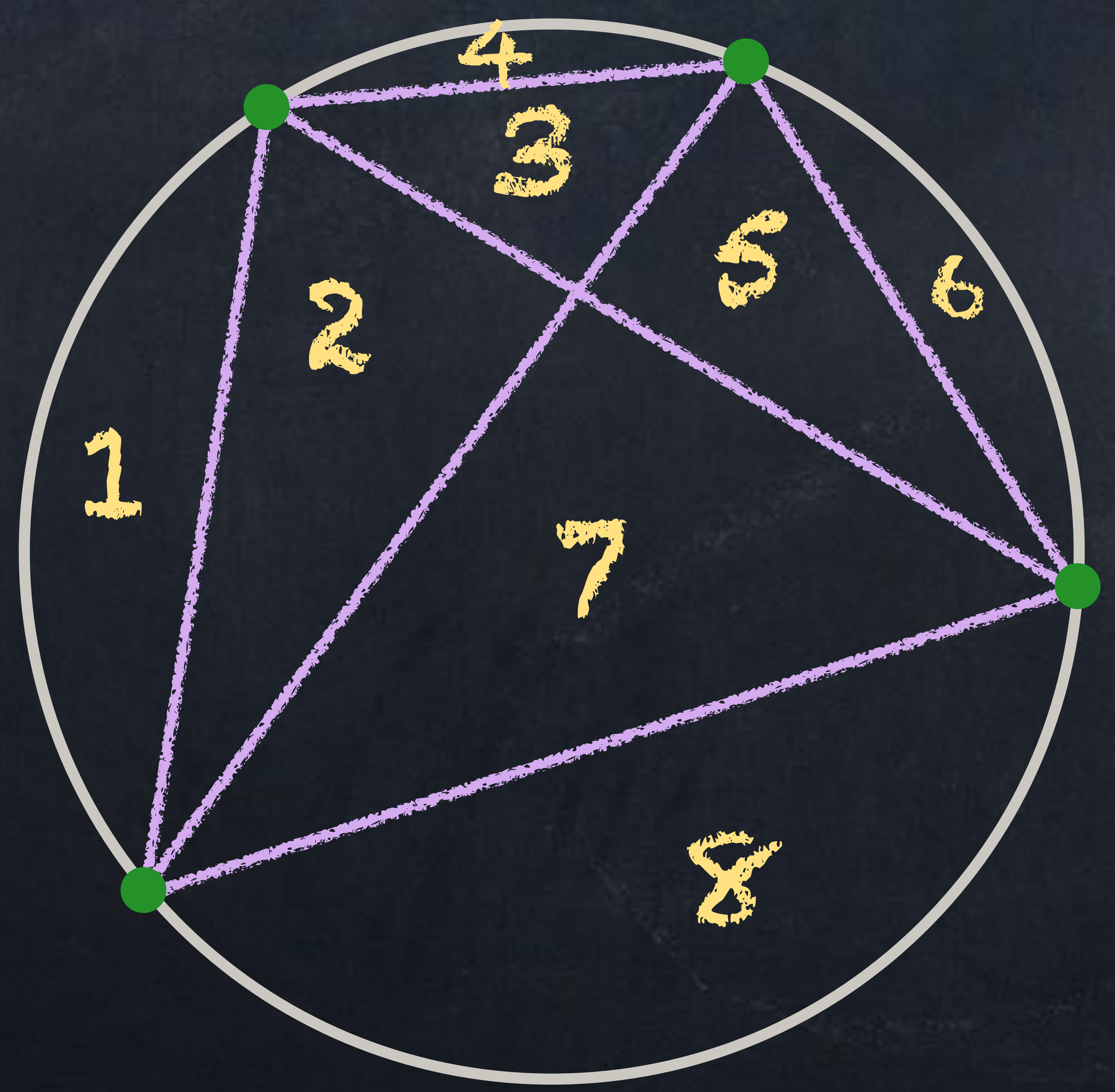
later

Patterns in math

Draw a 2 or more points on a circle.

Connect every pair of points with a straight line.

- How many lines did you draw?
- How many regions are in the circle?



4 Points

6 Lines

8 Regions

Patterns in math

N	1	2	3	4	5	6	7	8
L	0	1	3	6	10	15	21	28
R	1	2	4	8	16	31	57	99

Claim: $L = \dfrac{N(N-1)}{2}$ for all N.

Induction

For what values of N do we know for certain that N dots need $\frac{N(N-1)}{2}$ lines to connect them all?

- $N = 1$ ✅

Now suppose we knew that k dots need $\frac{k(k-1)}{2}$ lines for some k.

Then $k+1$ dots would need

$$\frac{k(k-1)}{2} + k = \frac{k^2-k}{2} + \frac{2k}{2} = \frac{k^2+k}{2} = \frac{(k+1)k}{2}$$

lines: just draw a line from the new dot to each of the k old dots.

Induction

For what values of N do we know for certain that N dots need $\frac{N(N-1)}{2}$ lines to connect them all?

- $N = 1$ ✅

IF we know our line formula is correct for k dots then we know that it is correct for $k + 1$ dots.

- We already know the formula is correct for $N = 1$, so now we know it is also correct for $N = 2$ ✅.
- Now we know $N = 2$, so we know $N = 3$ ✅ works too.
- And $N = 4$ ✅, etc.

Proofs

In graduate-level or professional mathematics, we always PROVE claims, that is, we explain exactly why they are true (like we just did for the line formula).

In this class, I will often just state a fact and ask you to believe it.

Binomial formula

You should know that $(a+b)^2 = a^2 + 2ab + b^2$.

You could also check that $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

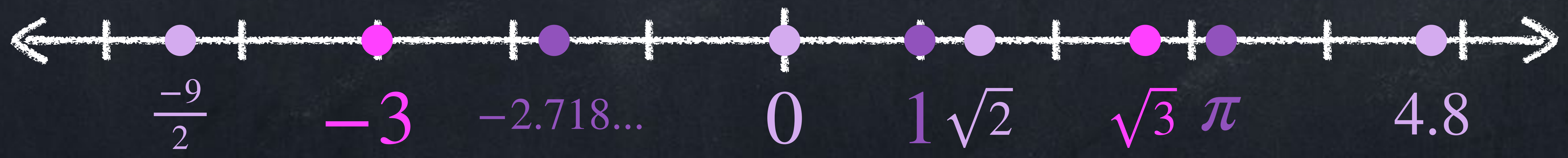
The Binomial Theorem says that for any whole power n we get

$$(a+b)^n = \frac{n}{1}a^n + \frac{n(n-1)}{2}a^{n-1}b + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-2}b^2 + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}a^{n-3}b^3 + \cdots + nb^n$$

It is possible to prove this (by induction or other explanations), but we will not.

Types of numbers

- **Natural** numbers: $0, 1, 2, 3, 4, \ldots$
  - In some books, only $1, 2, 3, 4, \ldots$
- **Integers**: $\ldots, -3, -2, -1, 0, 1, 2, 3, 4, \ldots$
- **Rational** numbers are all the numbers that *can* be written as one integer divided by another. Examples: $\frac{1}{2}, \frac{-2}{3}, 1.5, 8, 0, \frac{-5}{4}$
- **Real** numbers are all the values on a number line. Examples:



- **Complex** numbers - more on these later.

Solving a single equation

- If $2x = 18$, what must be the value of x?
  - Another way to ask this is “Solve $2x = 18$ for x.”
  - Answer: $x = 9$
- If $2x^2 = 18$, what possible values can x have?
  - Another way to ask this is “Solve $2x^2 = 18$ for x.”
  - Answer: $x = 3 \text{ or } x = -3$
- More examples:
  - Solve $3 = 7 - x$ for x. $x = 4$
  - Solve $2y = 18$ for y. $y = 9$
  - Solve $x + 2y = 0$ for y. $y = -x/2$

Solving quadratic equations

- Example 1: Solve $x^2 + 5x = 0$ for x.

Solving quadratic equations

- Example 2: Solve $x^2 + 9x - 6 = 2x$ for x.

The Quadratic Formula

The solutions to

$$ax^2 + bx + c = 0$$

(when $a \neq 0$) are

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

and

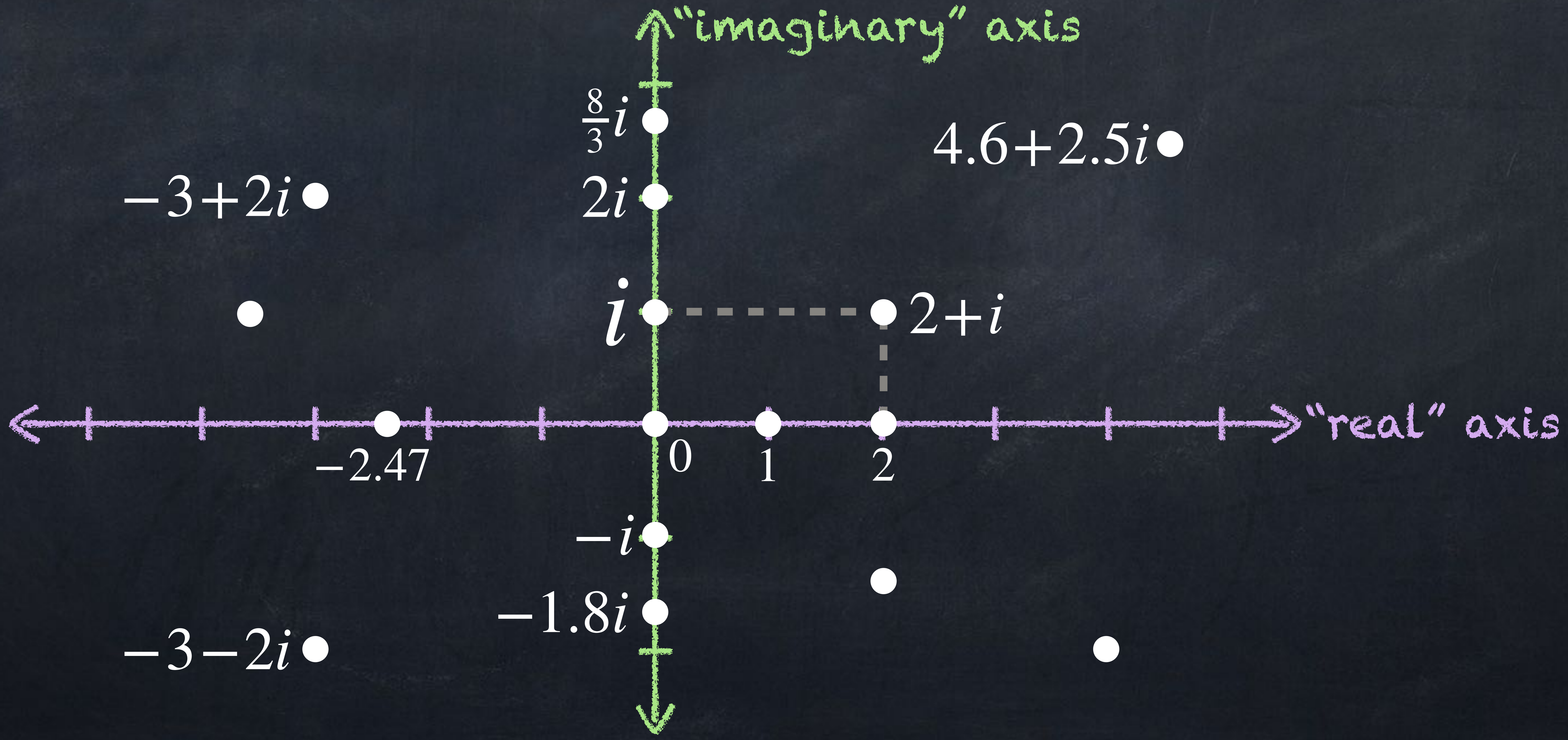
$$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Solving quadratic equations

- Example 3: Solve $x^2 + 5x + 12 = 0$ for x.

Complex numbers

- Algebra idea: allow square roots of negative numbers
- Geometry idea: 2D number plane



Complex numbers

- We call the horizontal (left/right) part of a complex number its **real part**, and we call the vertical (up/down) part its **imaginary part**.
- We write "$\mathrm{Re}\, z$" or "$\mathrm{Re}(z)$" for the real part of the complex number z. We write "$\mathrm{Im}\, z$" or "$\mathrm{Im}(z)$" for the imaginary part of z.
  Example:

  $$\mathrm{Re}(5-7i) = 5 \qquad \mathrm{Im}(5-7i) = -7$$

- If your complex numbers are in the form $\_ + \_i$, then addition is easy.

  $$(a+bi)+(c+di) = (a+b)+(c+d)i$$

  Example:

  $$(5-7i)+(9+4i) = (5+9)+(-7+4)i = 14-3i$$

Multiplication

- What does 5×3 mean?
  - 
  - More advanced: no pictures, just $5 + 5 + 5$.
- What does $5 \times \frac{1}{3}$ mean? 5×9.2 ? $7.65 \times (-12)$?
- We have changed the meaning of multiplication many times already.
- What does $(3 + 4i) \times i$ mean?

Multiplication

From now on, we will say that

$$i \times i = -1.$$

There are many good reasons for this, but for now just consider it a new part of the definition of how multiplication works.

People often write "$i = \sqrt{-1}$".

Algebra with complex #s

Using $i^2 = -1$ and standard algebra rules, we can can now do lots of computations with complex numbers.

$$5(3+7) = (5 \cdot 3) + (5 \cdot 7)$$

$$a(3+x) = (a \cdot 3) + ax$$

$$\begin{aligned} i(3+2i) &= i \times 3 \;\; + \;\; i \times 2i \\ &= 3i + 2(i \times i) \\ &= -2 + 3i \end{aligned}$$

Algebra with complex #s

Using $i^2 = -1$ and standard algebra rules, we can can now do lots of computations with complex numbers.

$$(5-i)(2+4i) = (5)(2) + (5)(4i) + (-i)(2) + (-i)(4i)$$

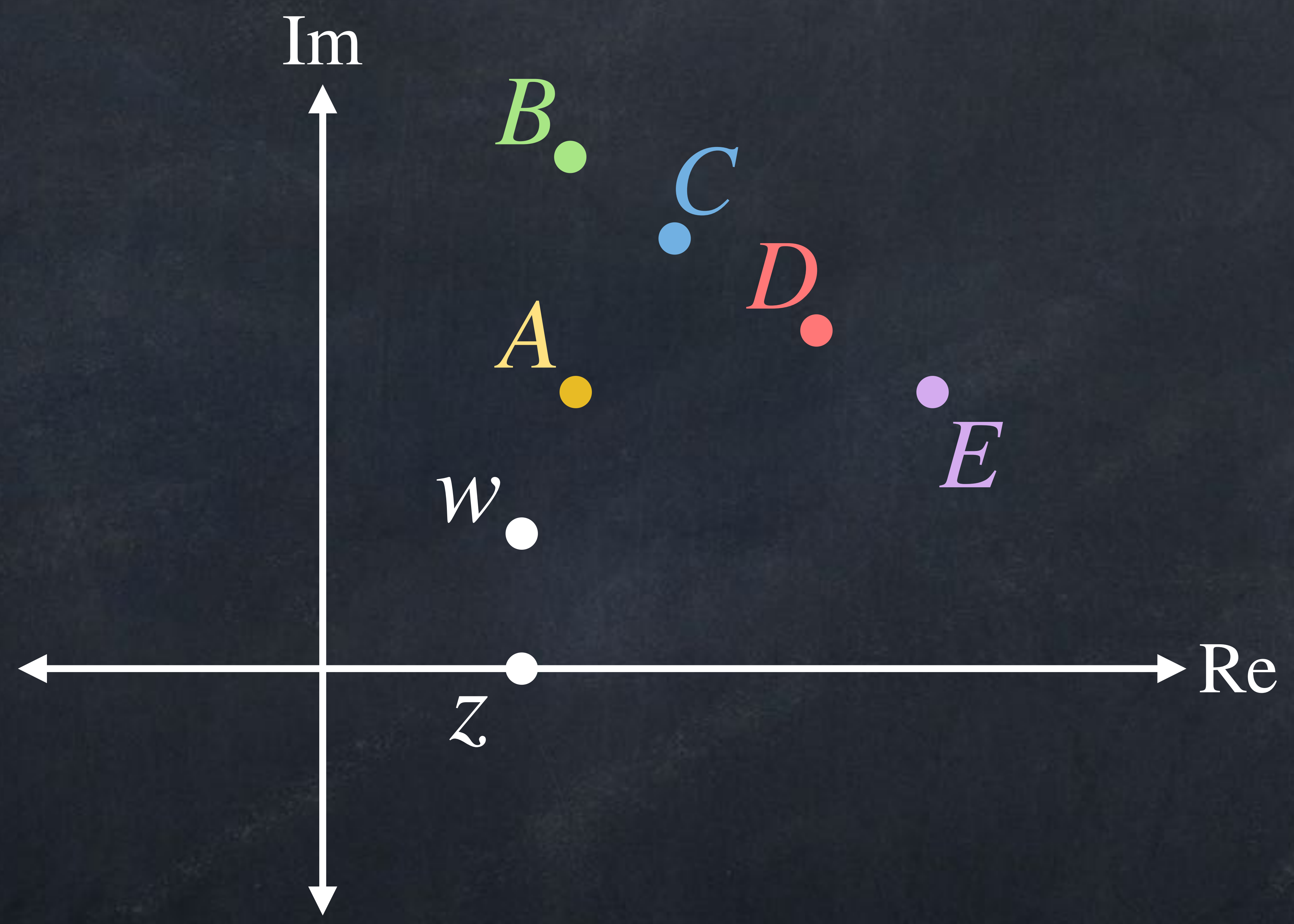
First Outside Inside Last

$$= 10 + 20i - 2i + 4$$
$$= 14 + 18i$$

- If you don't know how to expand $(a+b)(c+d)$ this way, you can use a slower method. But it's good to learn "FOIL".

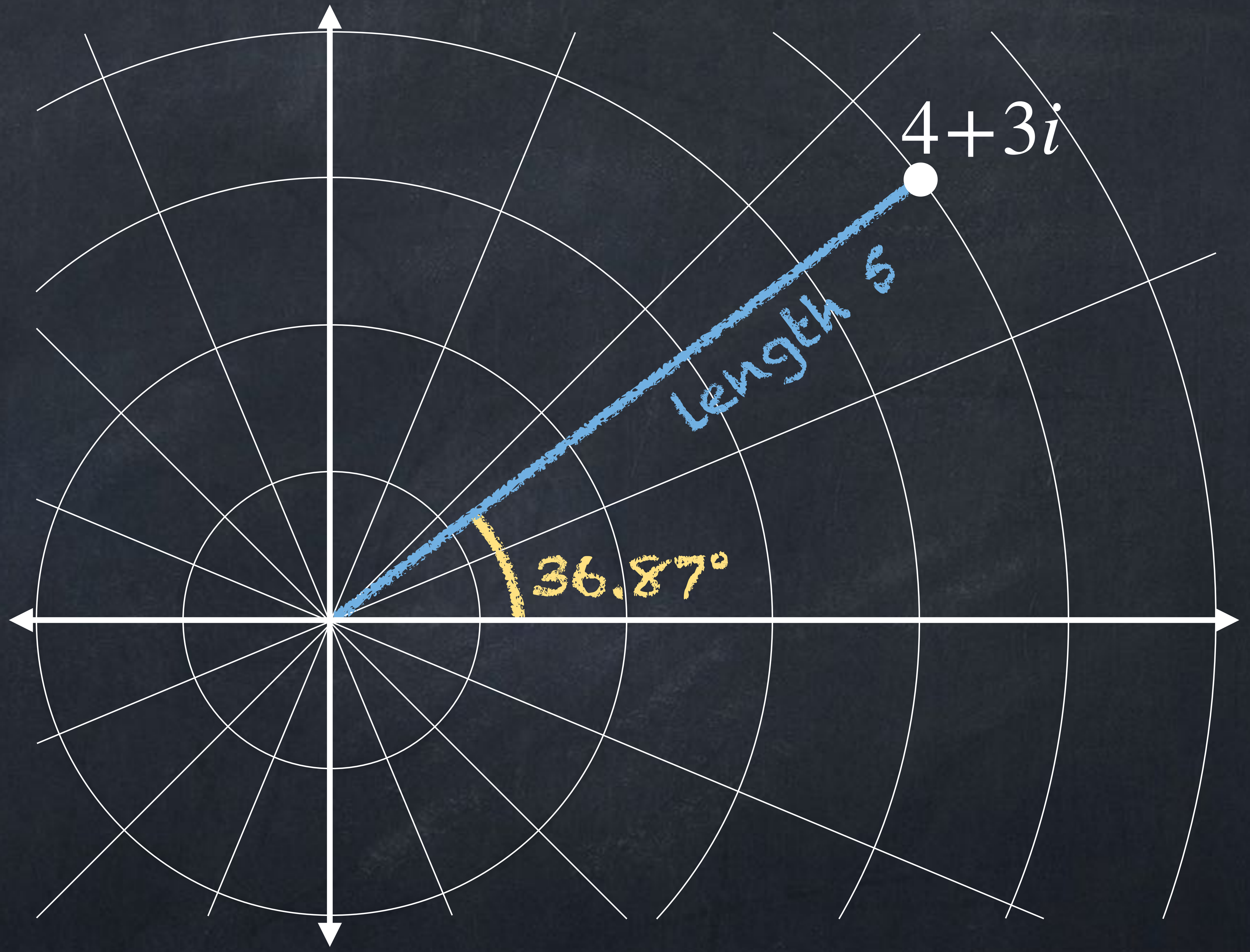
Calculators

- Which of the colored points is $z \cdot w$?



- A calculator does not help.
- We will answer this later.

Polar form



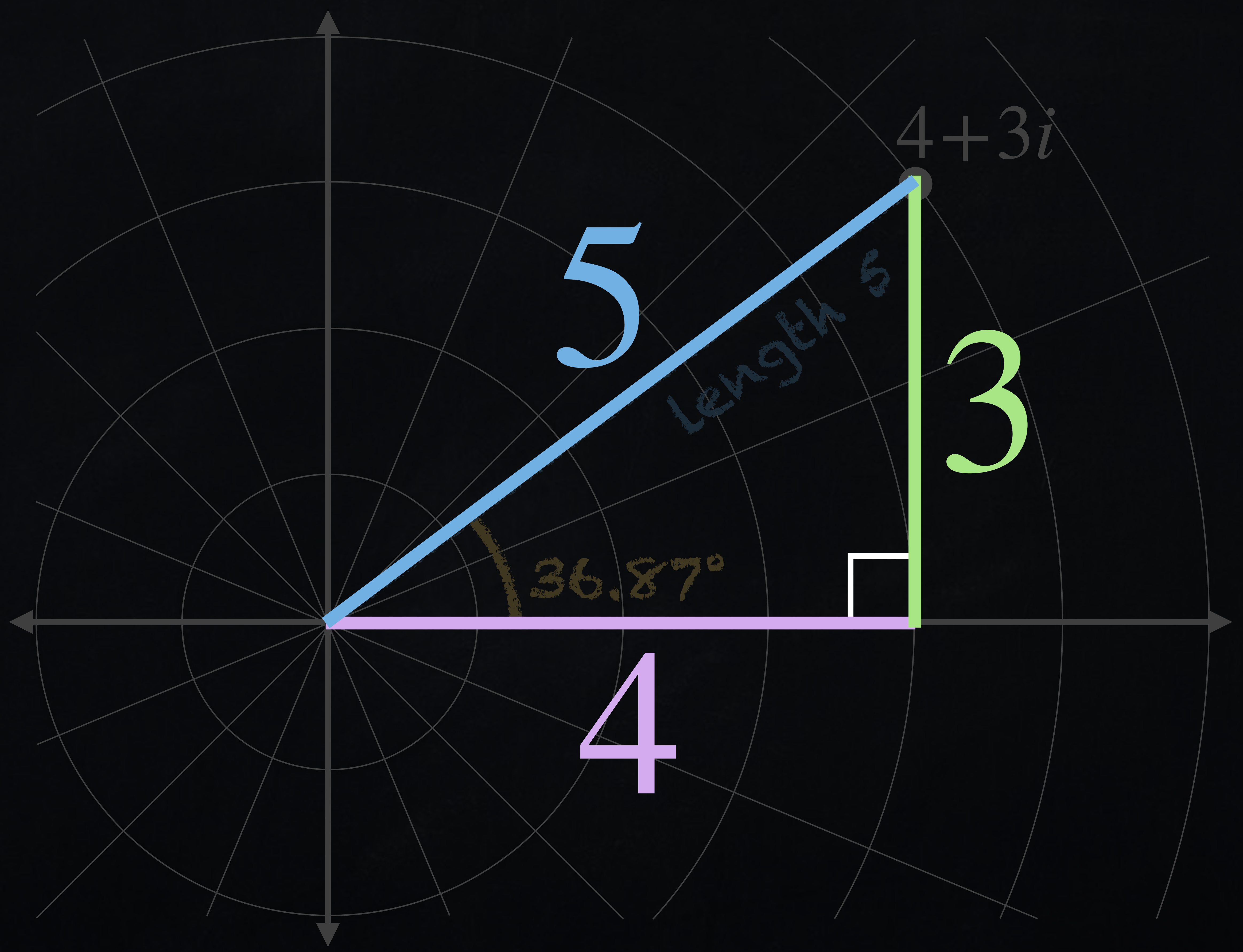
- Instead of 4 right and 3 up, we can describe this point as being 5 units away at an angle of 36.87°.

Polar form

The Pythagorean Theorem

If a and b are lengths of two sides of a right triangle and c is its hypotenuse, then

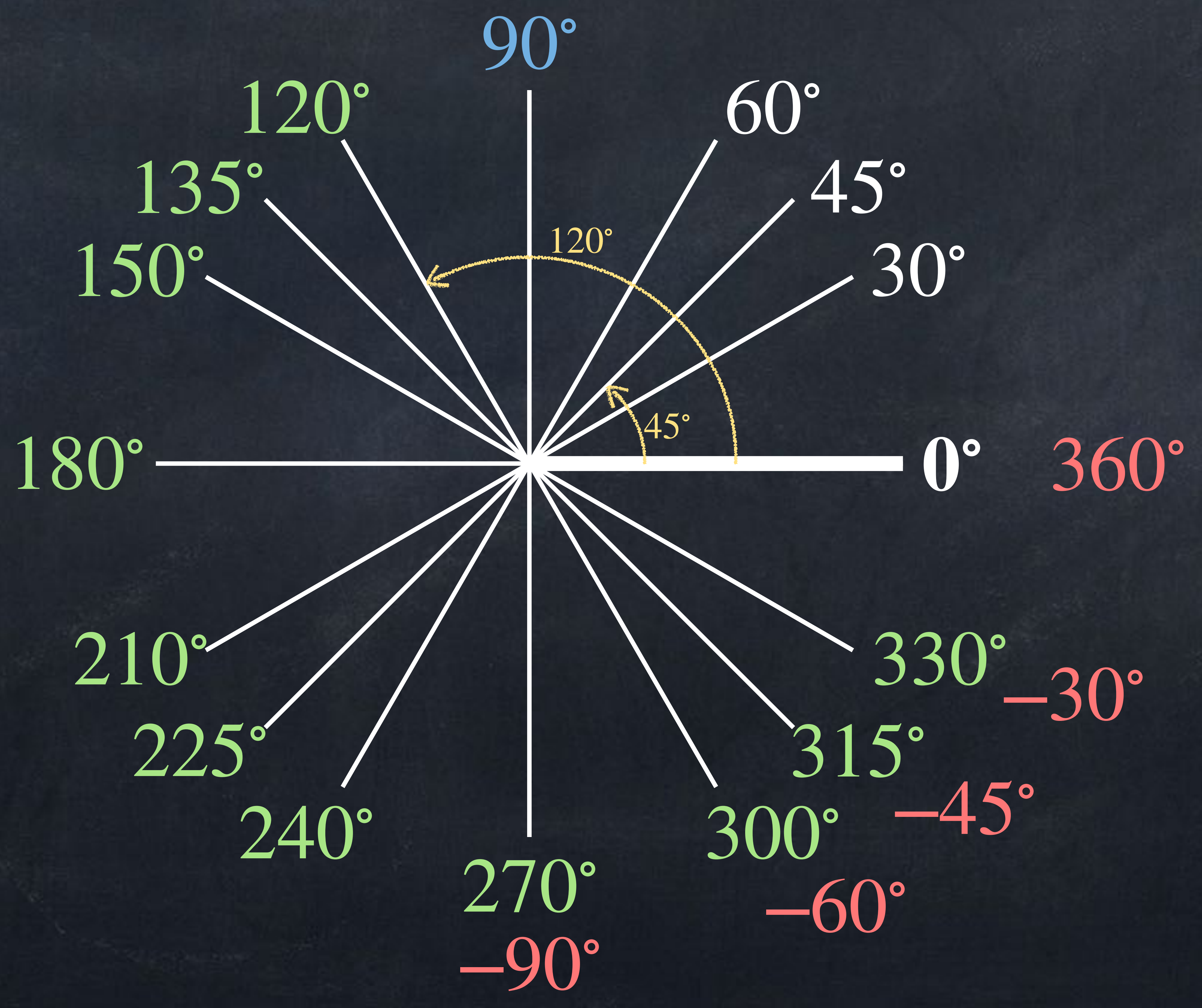
$$a^2 + b^2 = c^2.$$



- Instead of 4 right and 3 up, we can describe this point as being 5 units away at an angle of 36.87°.

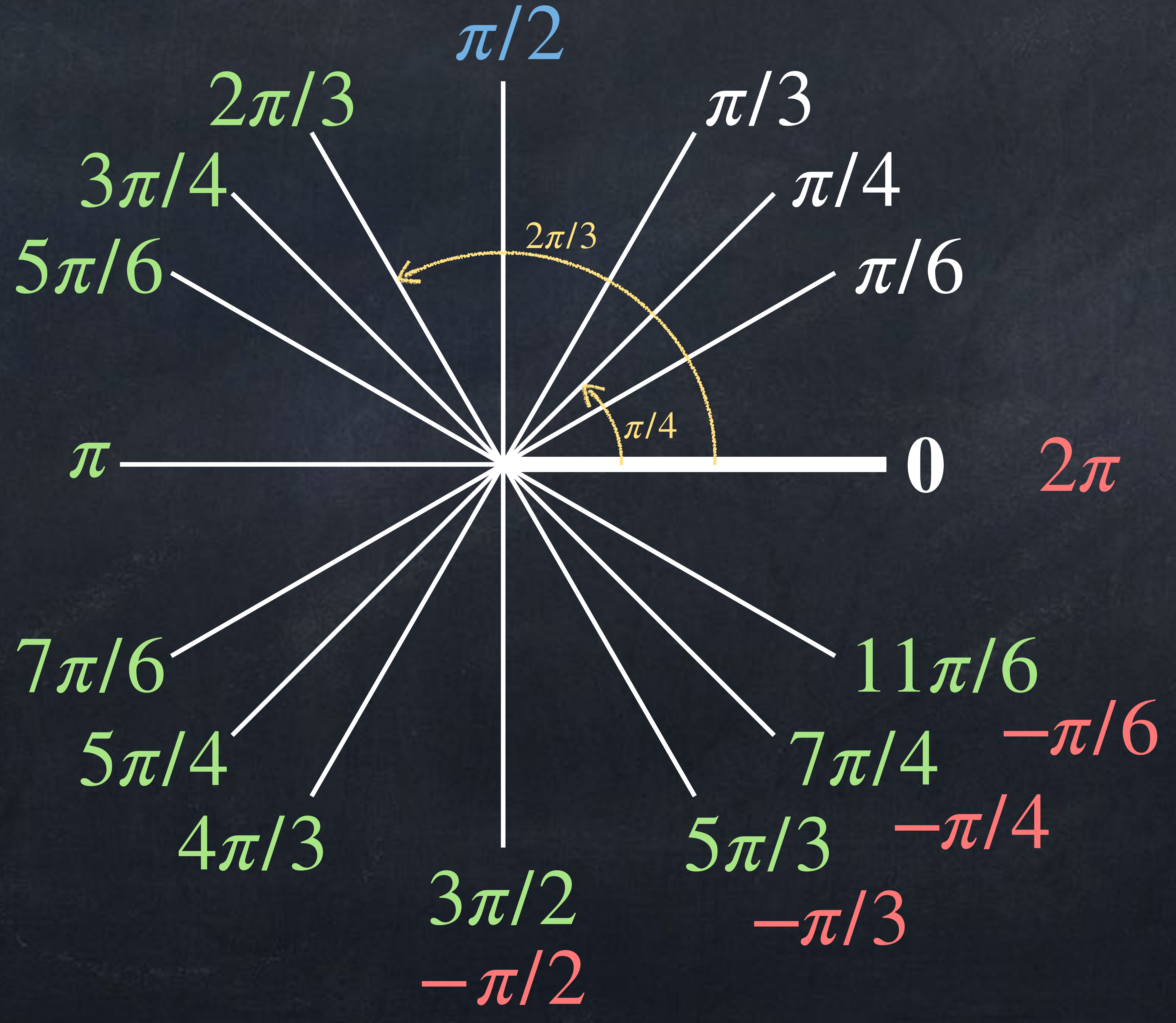
Measuring angles

- By default, 0° points to the right, and angles are measured counter-clockwise from there.

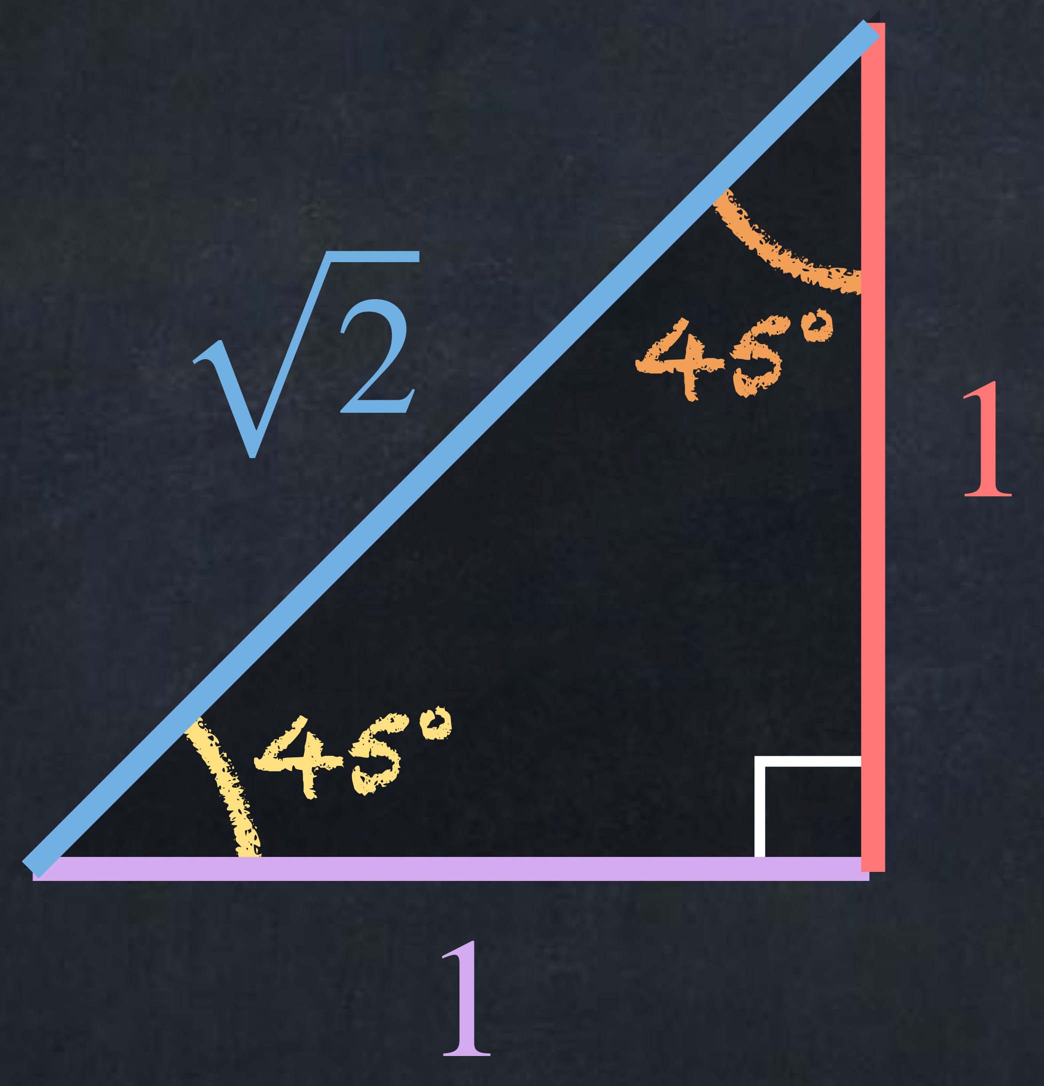


Measuring angles

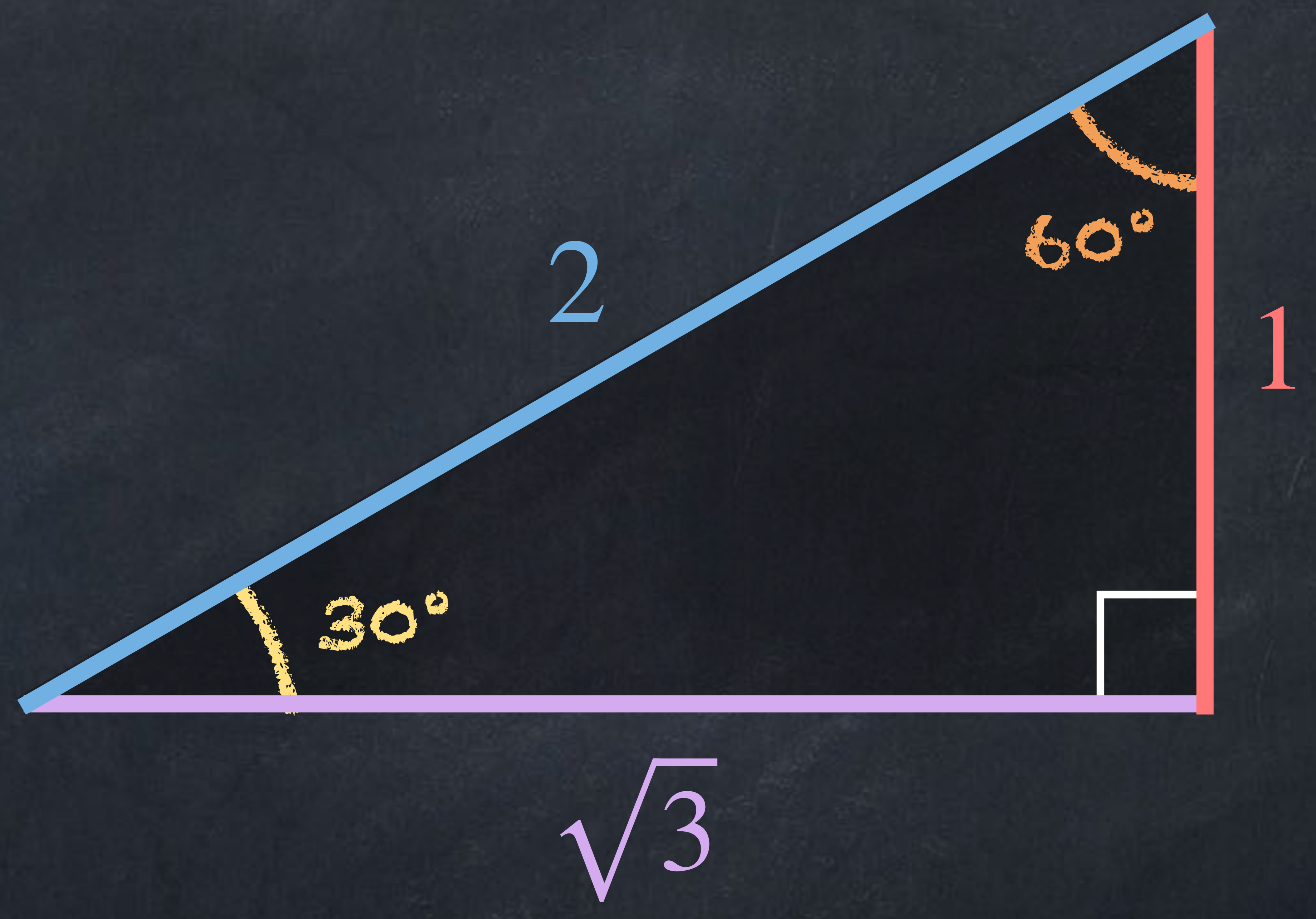
- Here is the same picture using radians.



Right triangles

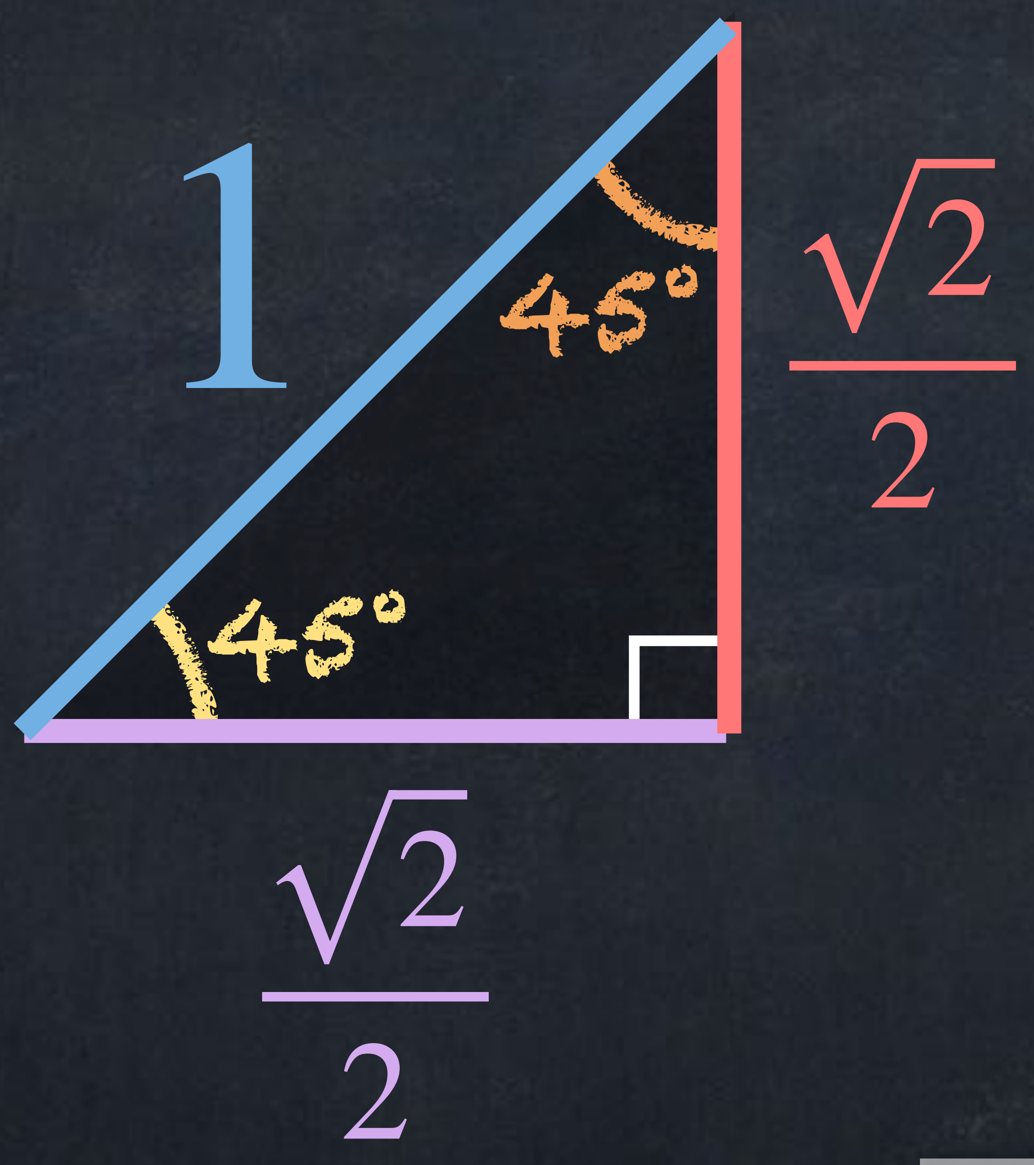


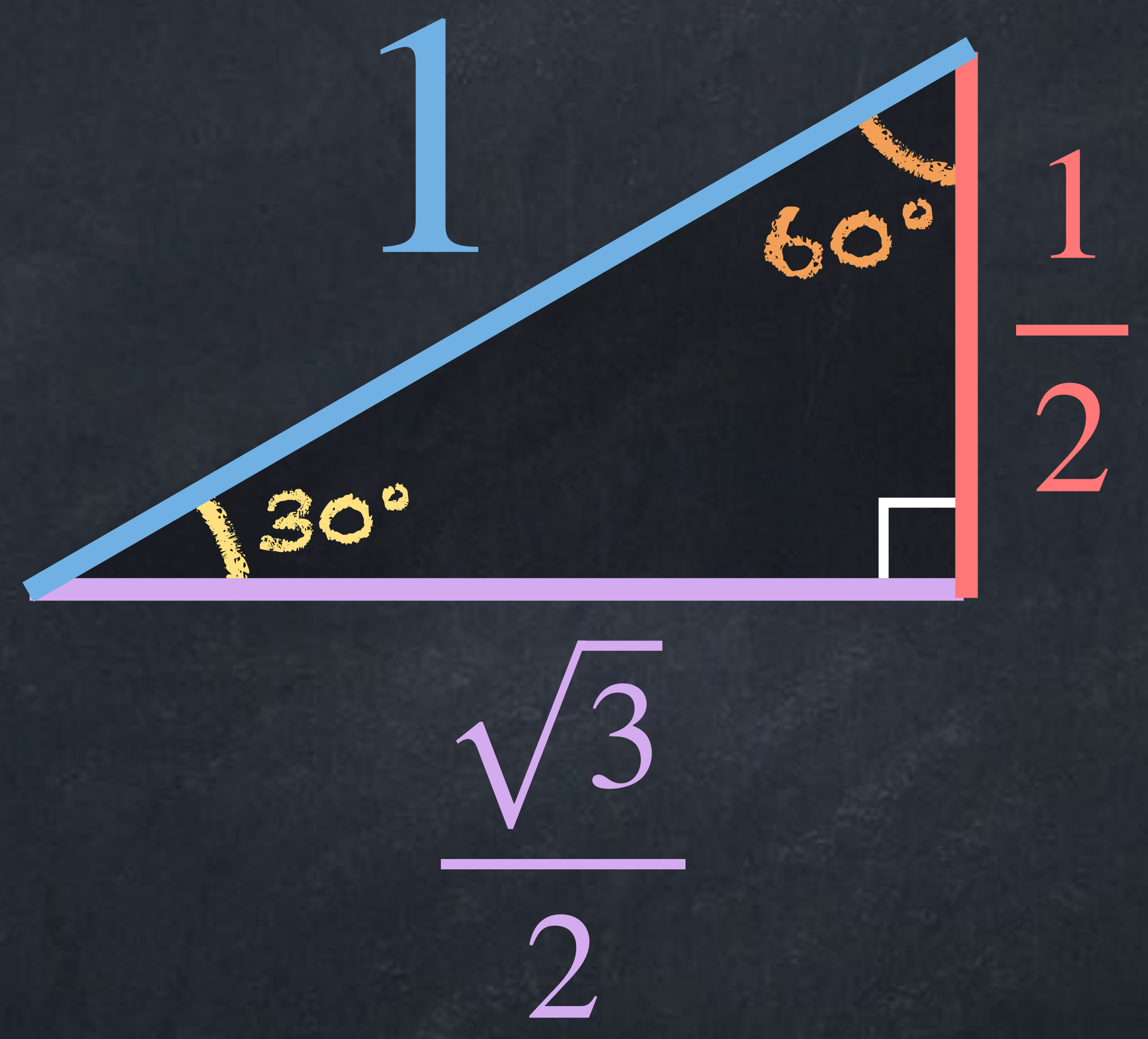
"45-45-90 triangle"



"30-60-90 triangle"

Right triangles



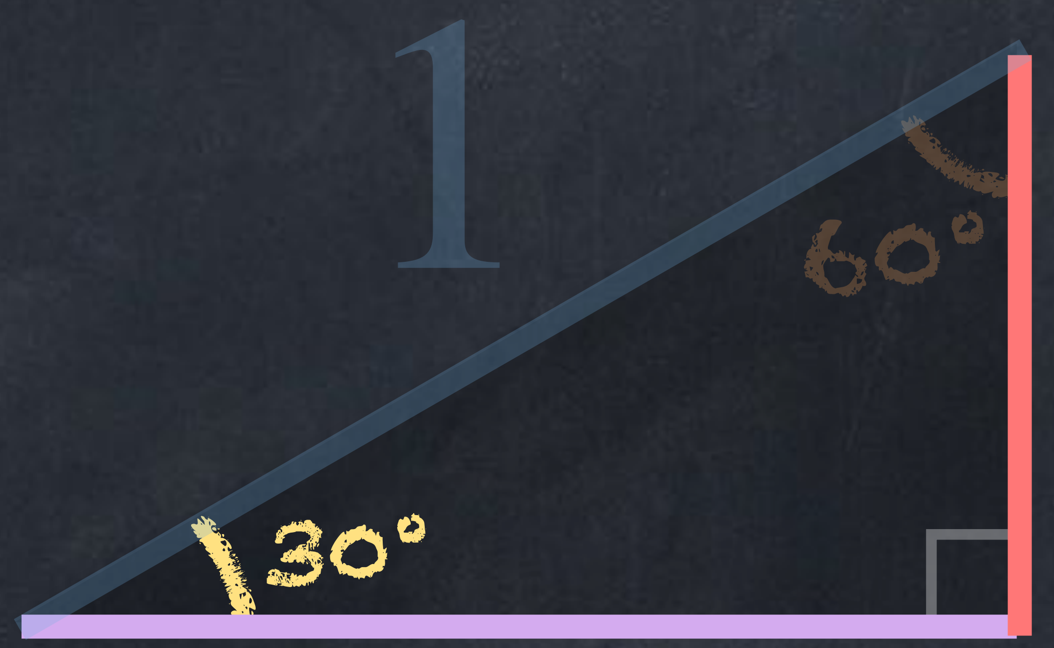


Memorize these numbers!

Right triangles

$$\frac{\sqrt{2}}{2} = \sin(45°)$$

$$\frac{\sqrt{2}}{2} = \cos(45°)$$



$$\frac{1}{2} = \sin(30°)$$

$$\frac{\sqrt{3}}{2} = \cos(30°)$$

Memorize these numbers!

Modulus and argument

The **modulus** of a complex number is its distance from 0.
We write $|z|$ for the modulus of a complex number z.

Examples:

- The modulus of $4+3i$ is 5.
- $|4+3i| = 5$
- $|2-7i| = \sqrt{53}$
- $|a+bi| = \sqrt{a^2+b^2}$ if a and b are real
- $|-8| = 8$

Modulus and argument

The **argument** of a complex number is the angle between the positive real axis and the line from 0 to that complex number.

We write $\arg(z)$ for the modulus of a complex number z.

Examples:

- The argument of $1+i$ is $45°$.
- $\arg(1+i) = 45°$
- The argument of $4+3i$ is $\arctan(\frac{3}{4})$, also written $\operatorname{atan}(\frac{3}{4})$ or $\tan^{-1}(\frac{3}{4})$. A calculator can tell us this is approximately 0.6435, or $36.89°$.