Math 1688

16 December 2021

Warm-up:
Dot products.

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Question 1

Compute the dot product

$[8, 0] \cdot [1, -2] =$ 8

Question 2

Compute the dot product

$[3, 6] \cdot \begin{bmatrix} 1 \\ -2 \end{bmatrix} =$ -9

Question 3

Compute the dot product

$[3, 6] \cdot \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} =$

impossible

Dimensions

The **dimension** of a vector (list) is how many numbers are in the list.

- The dimension of the vector $\begin{bmatrix} -4 \\ 9 \end{bmatrix}$ is 2.
- The dimension of the vector $\begin{bmatrix} 57 \\ 0 \\ {}^{1}/_{2} \end{bmatrix}$ is 3.

In order to add, subtract, or take dot products of vectors, they must have the same dimension.

Vector

A vector is…

- …a list of numbers.
- …an arrow.
- …a point.

$$\langle 1, 2 \rangle \qquad \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Matrix

A **matrix** is…

- …a rectangle of numbers.
- …a list of vectors.
- …(other ways to think about matrices will come later).

$$\begin{bmatrix} 5 & 8 \\ -2 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 4 & -4 \\ 2 & 12 \end{bmatrix} \qquad \begin{bmatrix} -8 & 19 & 4 & 4 \\ 19 & -4 & 2 & 6 \\ 15 & 8 & 2 & 16 \\ 3 & 14 & 0 & 12 \end{bmatrix}$$

Matrix

One **matrix** (“may-tricks” [meɪtrɪks]), two **matrices** (“may-trih-sees” [meɪtrɪsiz]).

We usually use a capital letter (no $\to$ or other mark) for a matrix variable.

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 3 & -5 & -10 \end{bmatrix}, \qquad M = \begin{bmatrix} 5 & 0 \\ 0 & \frac{1}{5} \end{bmatrix}.$$

The entries in a matrix are sometimes given two subscripts:

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{bmatrix} \text{ or } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}.$$

Row, column, dimensions

In the matrix $\begin{bmatrix} -1 & 2 & 21 \\ 13 & -1 & -7 \\ 10 & -9 & 13 \end{bmatrix}$,

- the **rows** are $\begin{bmatrix} -1 & 2 & 21 \end{bmatrix}$ and $\begin{bmatrix} 13 & -1 & -7 \end{bmatrix}$ and $\begin{bmatrix} 10 & -9 & 13 \end{bmatrix}$.
- the **columns** are $\begin{bmatrix} -1 \\ 13 \\ 10 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \\ -9 \end{bmatrix}$ and $\begin{bmatrix} 21 \\ -7 \\ 13 \end{bmatrix}$.
- the **main diagonal** is $\begin{matrix} -1 & & \\ & -1 & \\ & & 13 \end{matrix}$.

Row, column, dimensions

In the matrix $\begin{bmatrix} -1 & 2 & 21 \\ 13 & -1 & -7 \\ 10 & -9 & 13 \end{bmatrix}$,

- the **rows** are $[-1 \quad 2 \quad 21]$ and $[13 \quad -1 \quad -7]$ and $[10 \quad -9 \quad 13]$.
- the **columns** are $\begin{bmatrix} -1 \\ 13 \\ 10 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \\ -9 \end{bmatrix}$ and $\begin{bmatrix} 21 \\ -7 \\ 13 \end{bmatrix}$.

-1

- the **main diagonal** is -1 .

13

Row, column, dimensions

In the matrix $\begin{bmatrix} -1 & 2 & 21 \\ 13 & -1 & -7 \\ 10 & -9 & 13 \end{bmatrix}$,

- the **rows** are $[-1 \quad 2 \quad 21]$ and $[13 \quad -1 \quad -7]$ and $[10 \quad -9 \quad 13]$.
- the **columns** are $\begin{bmatrix} -1 \\ 13 \\ 10 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \\ -9 \end{bmatrix}$ and $\begin{bmatrix} 21 \\ -7 \\ 13 \end{bmatrix}$.
- the **main diagonal** is $\begin{matrix} -1 & & \\ & -1 & \\ & & 13 \end{matrix}$.

Row, column, dimensions

The dimension of the vector $\begin{bmatrix} -4 \\ 9 \end{bmatrix}$ is 2.

(or 2x1 if we think of this as a matrix)

The dimension of the vector $\begin{bmatrix} 57 \\ 0 \\ {}^{1}/_{2} \end{bmatrix}$ is 3.

(or 3x1 if we think of this as a matrix)

The **dimensions** of the matrix $\begin{bmatrix} 8 & 5 & -1 \\ 0 & 4 & 4 \end{bmatrix}$ are 2×3 (aloud: “2 by 3”).

A **square matrix** has the same number of rows as columns, like $\begin{bmatrix} 0 & 2 \\ -1 & 8 \end{bmatrix}$.

Matrix calculations

- scalar multiplication $9 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = ?$
- addition/subtraction $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = ?$
- matrix times a vector $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 10 \\ 11 \end{bmatrix} = ?$ (note: no · or × required)
- matrix multiplication $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = ?$ (note: no · or × required)

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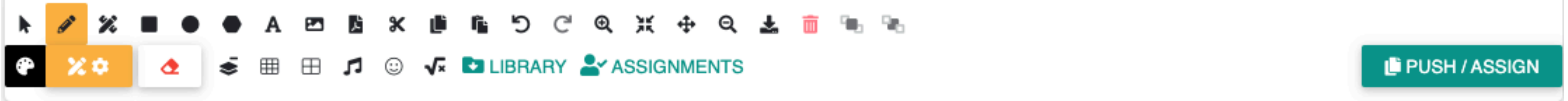
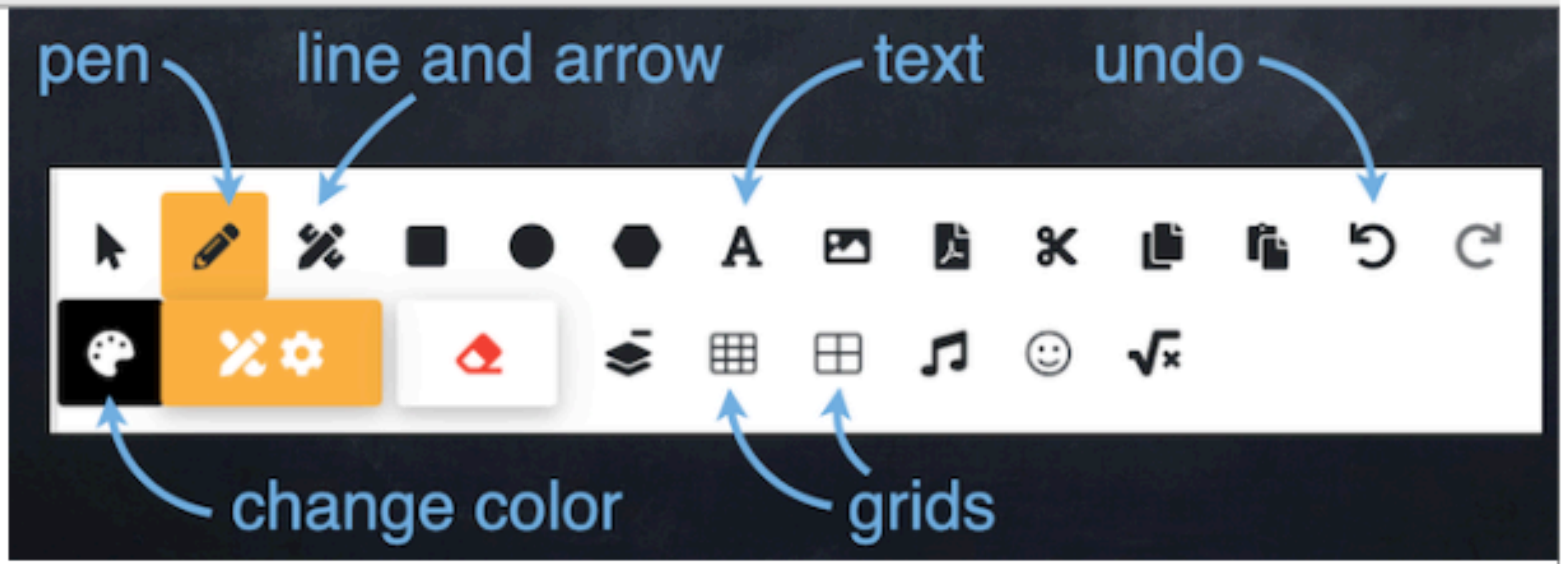


Rule: if $M = sA$ then $m_{ij} = s\,a_{ij}$.



Students, write or type your answer (fill in the four numbers):

$$\frac{1}{2}\begin{bmatrix}6 & 0\\ -8 & 1\end{bmatrix} = \begin{bmatrix}3 & 0\\ -4 & 1/2\end{bmatrix}$$

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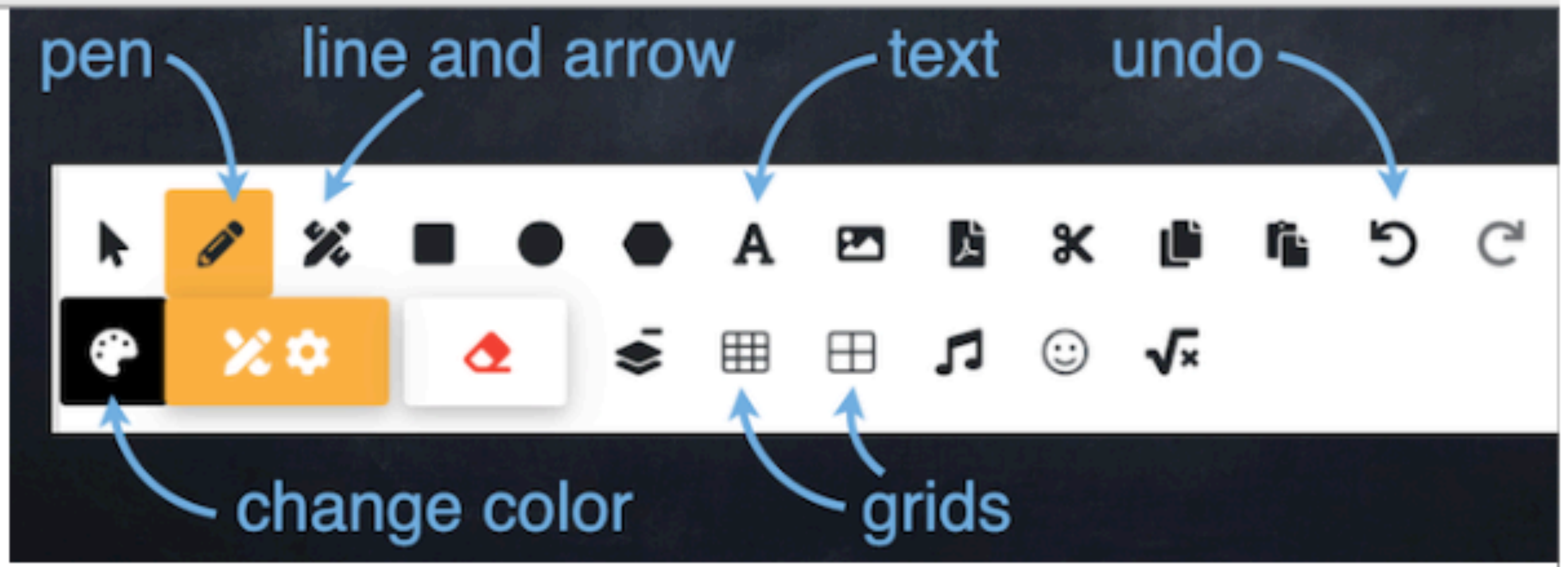
Rule: if $M = A+B$
then $m_{ij} = a_{ij} + b_{ij}$.



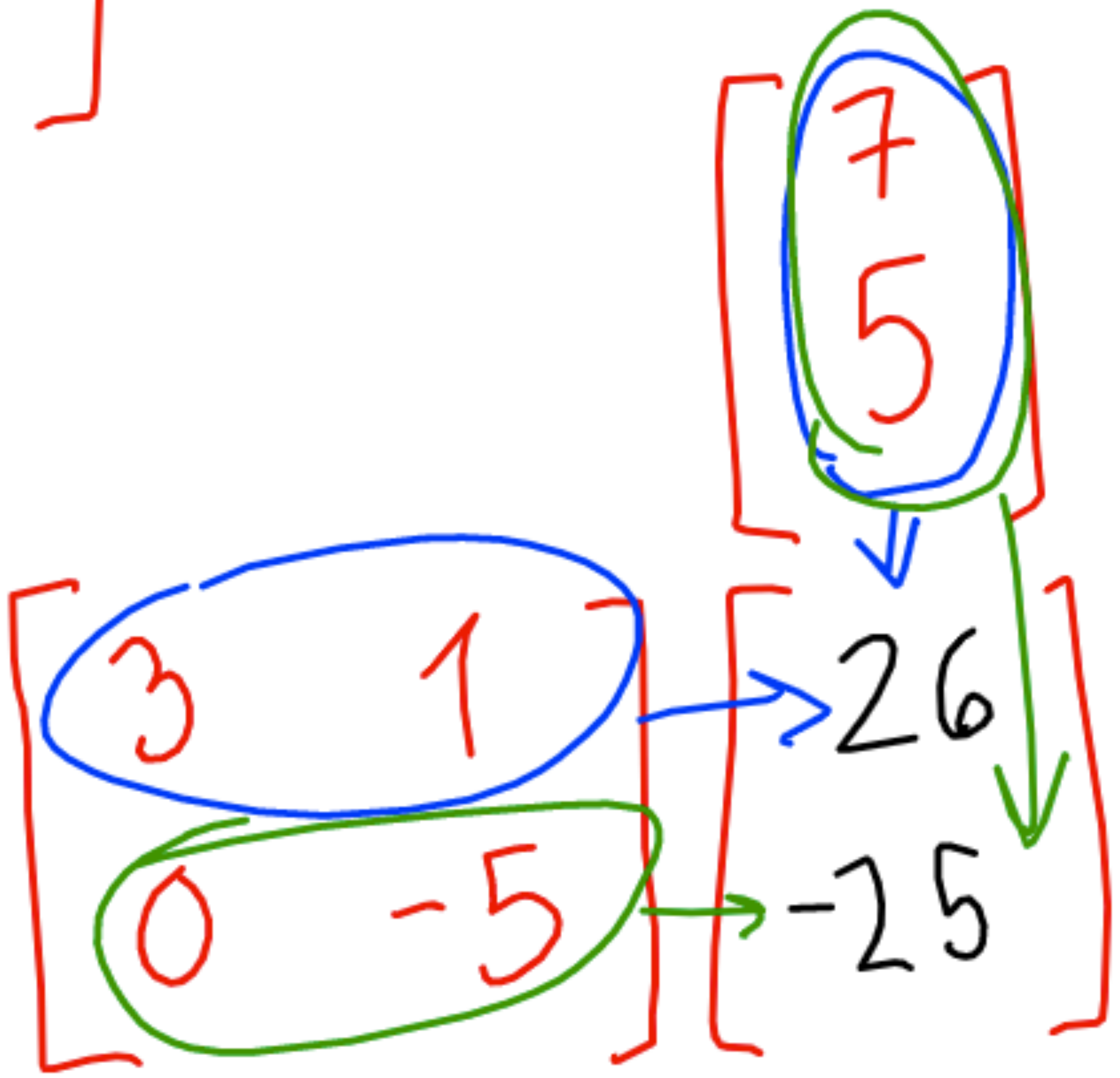
Students, fill in the six blanks.

$$\begin{bmatrix} 1 & 3 \\ 0 & 2 \\ -1 & 9 \end{bmatrix} + \begin{bmatrix} 2 & 4 \\ 6 & 7 \\ 6 & -4 \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 6 & 9 \\ 5 & 5 \end{bmatrix}$$

Rule: if $\vec{w} = A\vec{v}$ then
w_i = (row i of A) $\cdot$ $\vec{v}$.

dot product

$$\begin{bmatrix} 3 & 1 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} 7 \\ 5 \end{bmatrix} = \begin{bmatrix} 26 \\ -25 \end{bmatrix}$$



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Rule: if $\vec{w} = A\vec{v}$ then $\vec{w}_i$ = (row i of A) · $\vec{v}$.

dot product

★ $A\vec{v}$ is only possible if the # of columns of A = the dimension of $\vec{v}$.

Students, answer the following:

$$\begin{bmatrix} 3 & 6 \\ 8 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -9 \\ 8 \end{bmatrix}$$

$3 \cdot 1 + 6(-2)$

$8 \cdot 1 + 0(-2)$

Rule: if $\vec{w} = A\vec{v}$ then $\vec{w}_i$ = (row i of A) · $\vec{v}$.

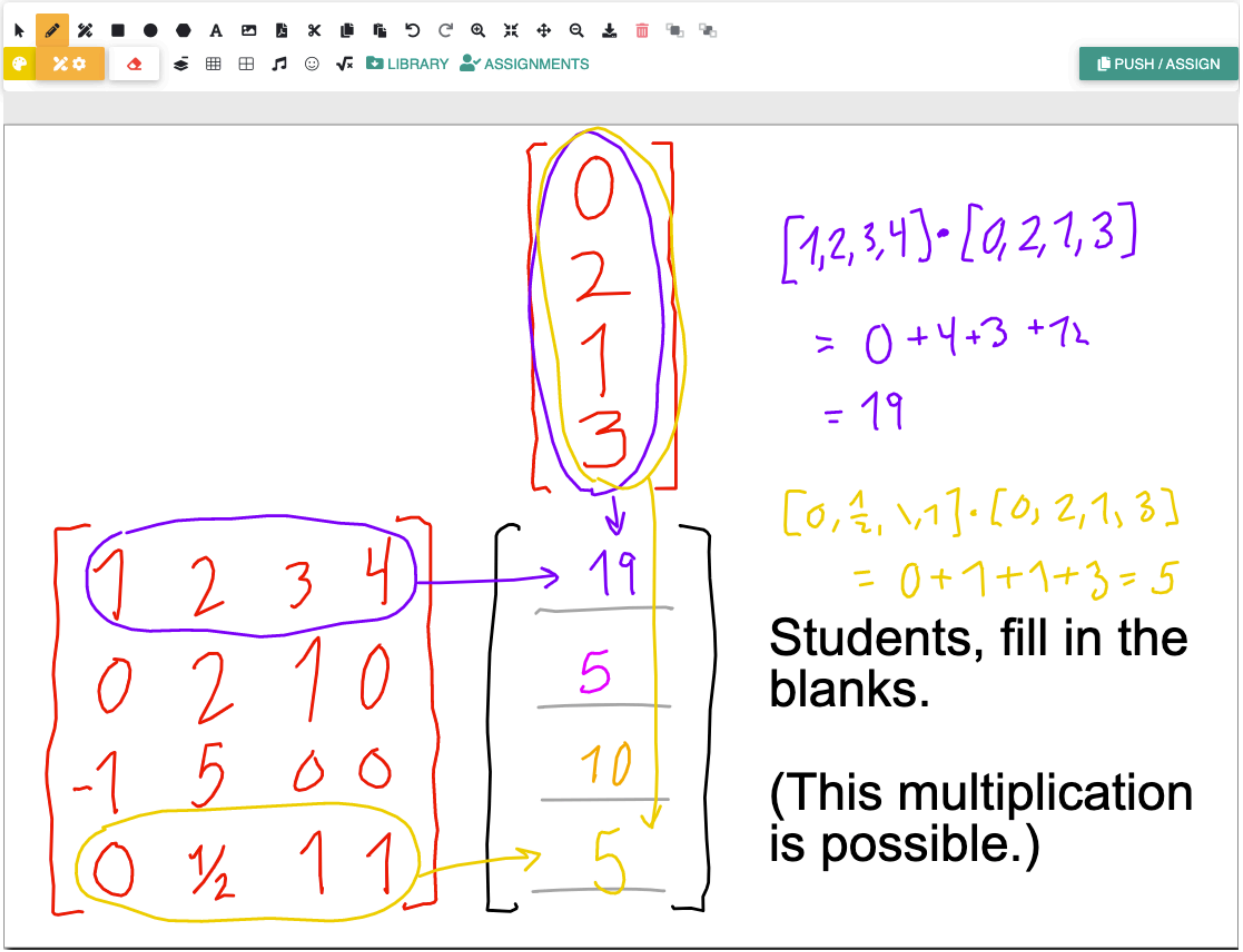
↑ dot product

★ $A\vec{v}$ is only possible if the # of columns of A = the dimension of $\vec{v}$.

Students, answer the following:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 0 \\ -1 & 5 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \\ \\ \end{bmatrix}$$

[1,2,3,4] · [0,2,1] is impossible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0
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1 2 3 4
0 2 1 0
-1 5 0 0
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5
[1,2,3,4]•[0,2,1,3]
= 0+4+3+12
= 19
[0, 1/2, 1, 1]•[0, 2, 1, 3]
= 0+1+1+3 = 5
Students, fill in the blanks.
(This multiplication is possible.)

Matrix calculations

- scalar multiplication $9\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix} = ?$
- addition/subtraction $\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix} + \begin{bmatrix}5 & 6\\7 & 8\end{bmatrix} = ?$
- matrix times a vector $\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}10\\11\end{bmatrix} = ?$ (note: no · or × required)
- matrix multiplication $\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}5 & 6\\7 & 8\end{bmatrix} = ?$ (note: no · or × required)

Rule: if $\vec{w} = A\vec{v}$ then w_i = (row i of A) $\cdot$ $\vec{v}$.

Rule: if $M = AB$ then m_{ij} = (row i of A) $\cdot$ (column j of B).

Example 1: $$\begin{bmatrix} 4 & 1 & 0 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 9 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 15 & 16 \end{bmatrix}$$

$$\begin{array}{ccc|cc} & & & 1 & -1 \\ & & & 2 & 9 \\ & & & 3 & 1 \\ \hline 4 & 1 & 0 & 6 & 5 \\ -2 & 1 & 5 & 15 & 16 \end{array}$$

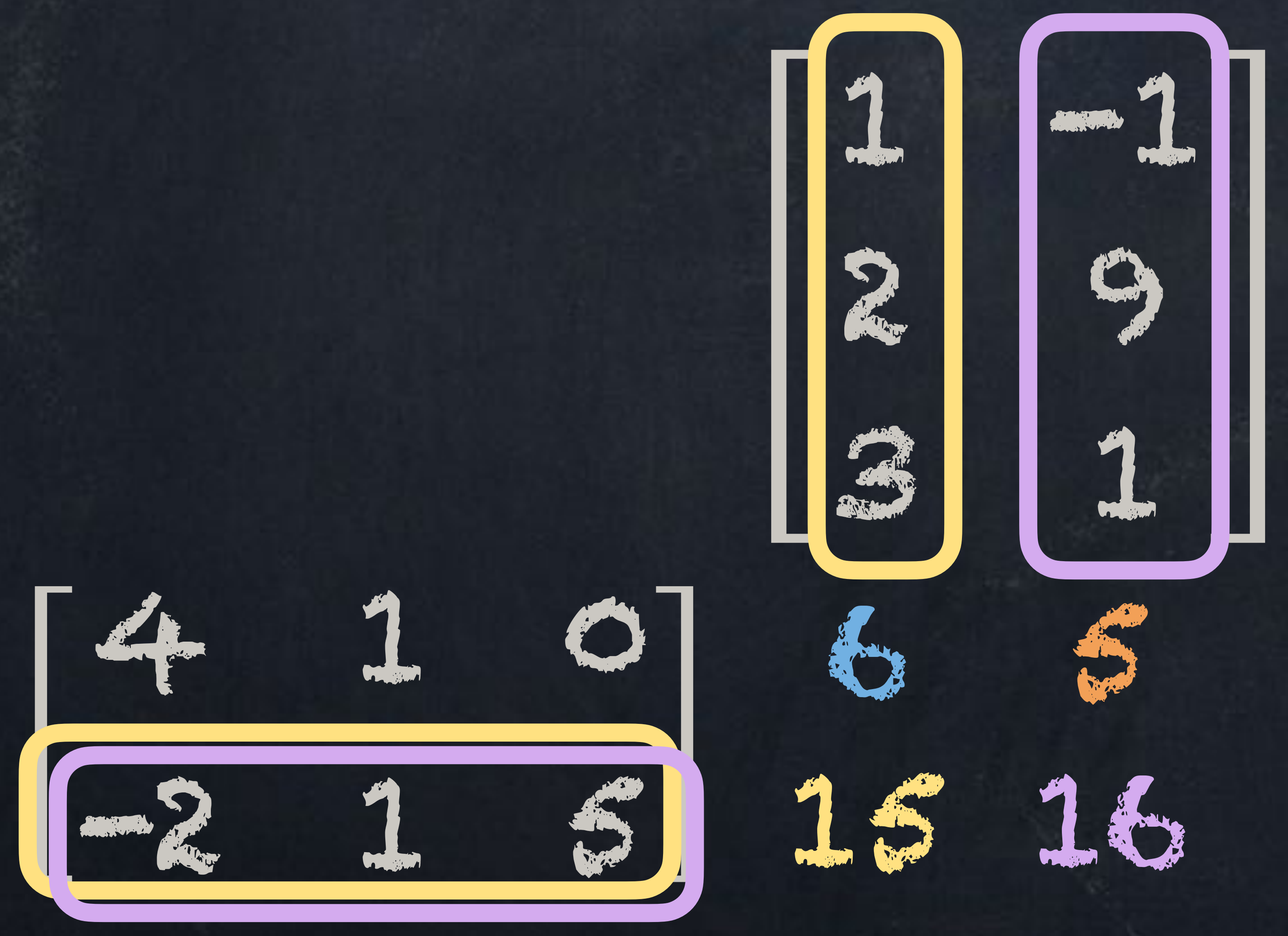
$$[4,1,0] \cdot [1,2,3] = 4+2+0 = 6$$

$$[4,1,0] \cdot [-1,9,1] = -4+9+0 = 5$$

Rule: if $\vec{w} = A\vec{v}$ then w_i = (row i of A) · $\vec{v}$.

Rule: if $M = AB$ then m_{ij} = (row i of A) · (column j of B).

Example 1: $$\begin{bmatrix}4 & 1 & 0\\ -2 & 1 & 5\end{bmatrix}\begin{bmatrix}1 & -1\\ 2 & 9\\ 3 & 1\end{bmatrix} = \begin{bmatrix}6 & 5\\ 15 & 16\end{bmatrix}$$



$$[-2,1,5]\cdot[1,2,3] = -2+2+15 = 15$$

$$[-2,1,5]\cdot[-1,9,1] = 2+9+5 = 16$$

Rule: if $M = AB$ then m_{ij} = (row i of A) · (column j of B).

Example 2: $\begin{bmatrix} 4 & 1 & 0 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 4 & 8 \\ 6 & -3 \end{bmatrix}$ impossible!

because [4,1,0] · [4,6] is impossible

2 x 3   2 x 2

The "inner" numbers must agree for AB to exist.

★ The matrix multiplication AB is only possible if (# of columns of A) = (# of rows of B).

Rule: if $M = AB$ then m_{ij} = (row i of A) · (column j of B).

Example 1: again

$$\begin{bmatrix} 4 & 1 & 0 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 9 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 15 & 16 \end{bmatrix}$$

2 x 3 3 x 2 2 x 2

The "inner" numbers must agree for AB to exist.

The "outer" numbers give the dimensions of AB.

Rule: if $M = AB$ then m_{ij} = (row i of A) · (column j of B).

Example 3: $$\begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 11 & 8 \\ 1 & 9 \end{bmatrix}$$

Example 4: $$\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 7 & 17 \\ 0 & 13 \end{bmatrix}$$

★ When multiplying matrices, AB and BA can be different!