Math 1688

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Warm-up:
System of equations.

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Systems of linear equations

A **linear equation** is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

where $x_1, \ldots, x_n$ are variables and $a_1, \ldots, a_n, b$ are coefficients (usually each a_i is a just a constant number, but it could be some expression that does not involve any x_i).

A **system of linear equations** (or just **system**) is a collection of linear equations with the same variables.

- Some equations may have coefficients of 0 for some variables, so you might not see every variable appear in every equation.
- Often, we will have the same number of variables as equations, but this is not necessary.

Systems of linear equations

Examples:

$$\begin{cases} 3x - 7y = 4 \\ x + 8y = 2 \end{cases}$$

$$\begin{cases} 3x - 7y = 4 \\ x + 8y = 2 \\ 2x + 5y = 7 \end{cases}$$

$$\begin{cases} 7x + 2y + 9z = -9 \\ 7x + 2y + 9z = 4 \\ -4x - 3y + 5z = 2 \end{cases}$$

$$\begin{cases} 7x + 2y + 9z = -9 \\ -4x - 3y + kz = 2 \end{cases}$$

$$\begin{cases} 7a + 2b + 9c = -9 \\ 8a + 6c = 0 \\ -4a - 3b + 5c = 2 \end{cases}$$

$$\begin{cases} 6x_1 + 2x_2 - 5x_3 + x_4 = 1 \\ 5x_1 - 7x_3 + 2x_4 = 3 \end{cases}$$

Systems of equations

Finding values or formulas for the variables in a system is called "solving" the system. Any assignment that makes all equations true is a **solution**.

Example: The only solution to

$$\begin{cases} a + b = 3 \\ a - b = 7 \end{cases}$$

is $(a, b) = (5, -2)$.

Example: $\begin{cases} a^2 + b = 3 \\ a^2 - b = 7 \end{cases}$ has two solutions:

$$(a, b) = (\sqrt{5}, 2) \text{ and } (a, b) = (-\sqrt{5}, 2).$$

not linear

There are *many* methods to solve systems of linear equations by hand. Some of the most common are

- Substitution
- Elimination
- Matrix inverse
- Cramer's Rule.

Of course, computers can solve equations for us.

Question: How many solutions can a *linear* system have?

2 equations and 2 variables

$$\begin{cases} 3x - 6y = 4 \\ x + 8y = 2 \end{cases}$$

$$\begin{cases} 3x - 6y = 4 \\ 3x - 6y = 2 \end{cases}$$

$$\begin{cases} 3x - 6y = 3 \\ x - 2y = 1 \end{cases}$$

3 equations and 3 variables

$$\begin{cases} 3x - 6y + z = 7 \\ x + 8y - z = 4 \\ 2x + 16y - 2z = 1 \end{cases}$$

$$\begin{cases} 3x - 6y + z = 7 \\ x + 8y - z = 4 \\ -x + y + 2z = 3 \end{cases}$$

Any linear systems—with any number of variables and any number of equations—will have either

- **0** solutions,
- exactly **1** solution, or
- **infinitely many** solutions.

For infinity many solutions with 2 variables or 3 variables, the collection of all solutions will form a line or a plane.

There are many methods to solve systems of linear equations by hand.

- Substitution
- Elimination

} Fewer calculations, but you have to be clever about what steps to take.

- Matrix inverse*
- Cramer's Rule*

} Follow the same steps every time, but do a lot of calculations.

It is also possible to determine the number of solutions—zero, one, or infinity—without actually solving the system.

- Determinant*
- Rank

* only when # of equations = # of variables

Substitution

Solving a system by **substitution**:

1. Re-write one equation as "$x_i = \dots$" for some i.
2. *Substitute* this expression for x_i into other equation and simplify.
3. Repeat 1 & 2 until you have a specific value for some x_j.
4. Find values for all other variables using previous equations.

This general description might be confusing, but with specific examples it's actually very simple.

- However, deciding which equations to use at each step and which variables to solve for (in what order) can be hard.
- Usually any order will work, but some will be slower or faster.

Substitution

Example: solve $\begin{cases} 5x - 2y = 15 \\ x + 4y = 14 \end{cases}$ using substitution.

Word problems

Two numbers have a sum of 12 and a difference of 4.

- What are the two numbers?

Word problems

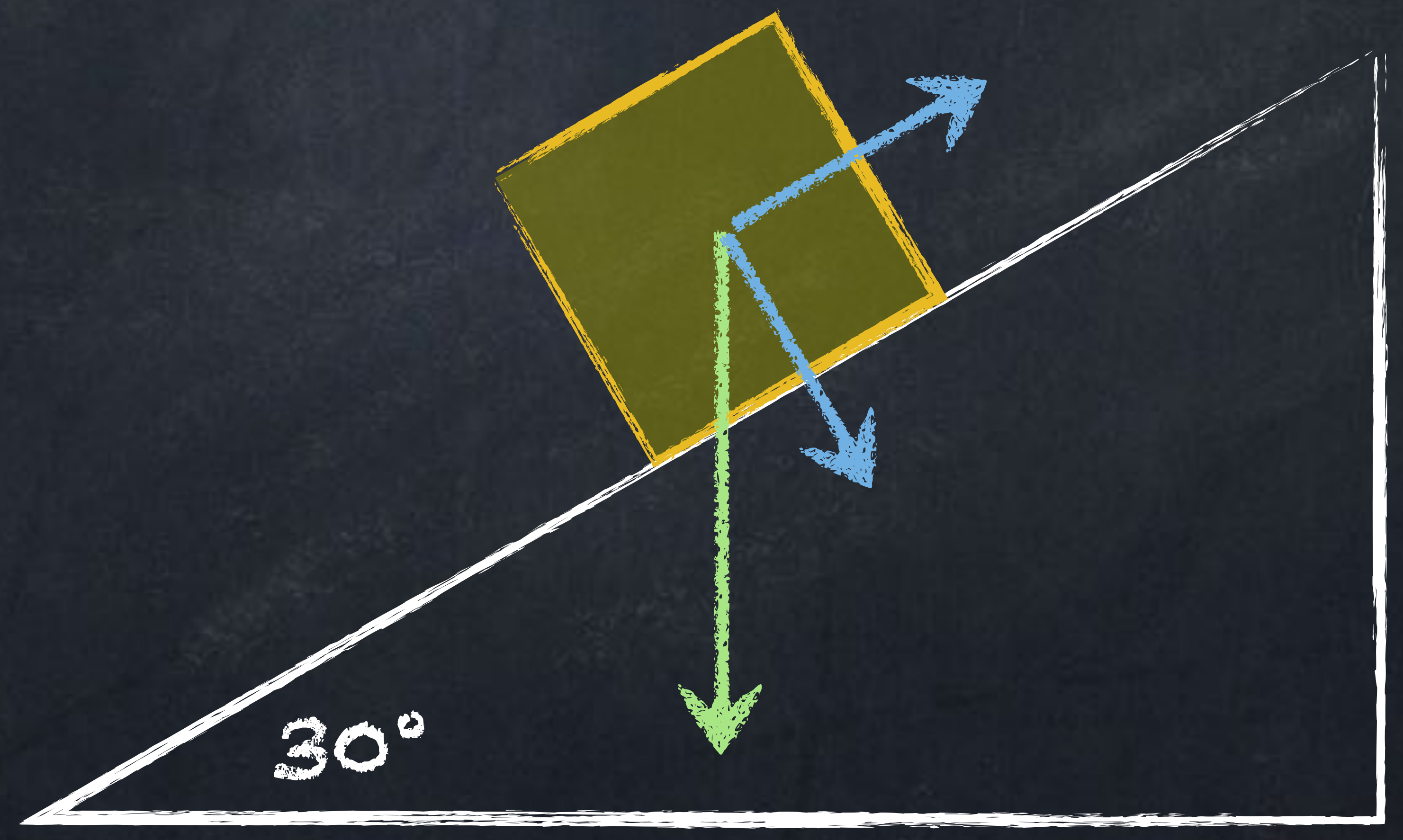
A store sells pizzas in three sizes: small, medium, and large. Buying one of each costs 108 zł total. Two smalls and one large costs 100 zł total, and buying one small and three mediums costs 144 zł total.

- How much does each size cost?

Word problems

For a 30° hill, the vector $[\sqrt{3}, 1]$ is parallel to the hill and the vector $[1, -\sqrt{3}]$ points directly into to the hill. A box on the hill is acted on by the force of gravity $\overrightarrow{F} = [0, -4]$.

- Write the force of gravity as a linear combination of the vectors along and into the hill.



Word problems

- Write $[0, -4]$ as a linear combination of $[\sqrt{3}, 1]$ and $[1, -\sqrt{3}]$.

System as a matrix equation

The system of three equations

$$\begin{cases} 6x + y + 5z = 5 \\ \phantom{6x+{}} 2y + 9z = 3 \\ -x + 4y + 18z = 5 \end{cases}$$

can be written as the single equation

$$\begin{bmatrix} 6 & 1 & 5 \\ 0 & 2 & 9 \\ -1 & 4 & 18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 5 \end{bmatrix}$$

using matrices.

We usually write this as $AX = B$ and call A the “matrix of coefficients”.