Math 1688

Tuesday, 4 January

Warm-up: Inverses.

theadamabrams.com/live

Schedule

4.1 Class (today)

13.1 Class Group B: lecture, then midterm exam (45 min)

20.1 Class

27.1 Class

9.2 Final exam (90 min)

16.2 Final exam (90 min) – 2nd attempt

Online quizzes/exams

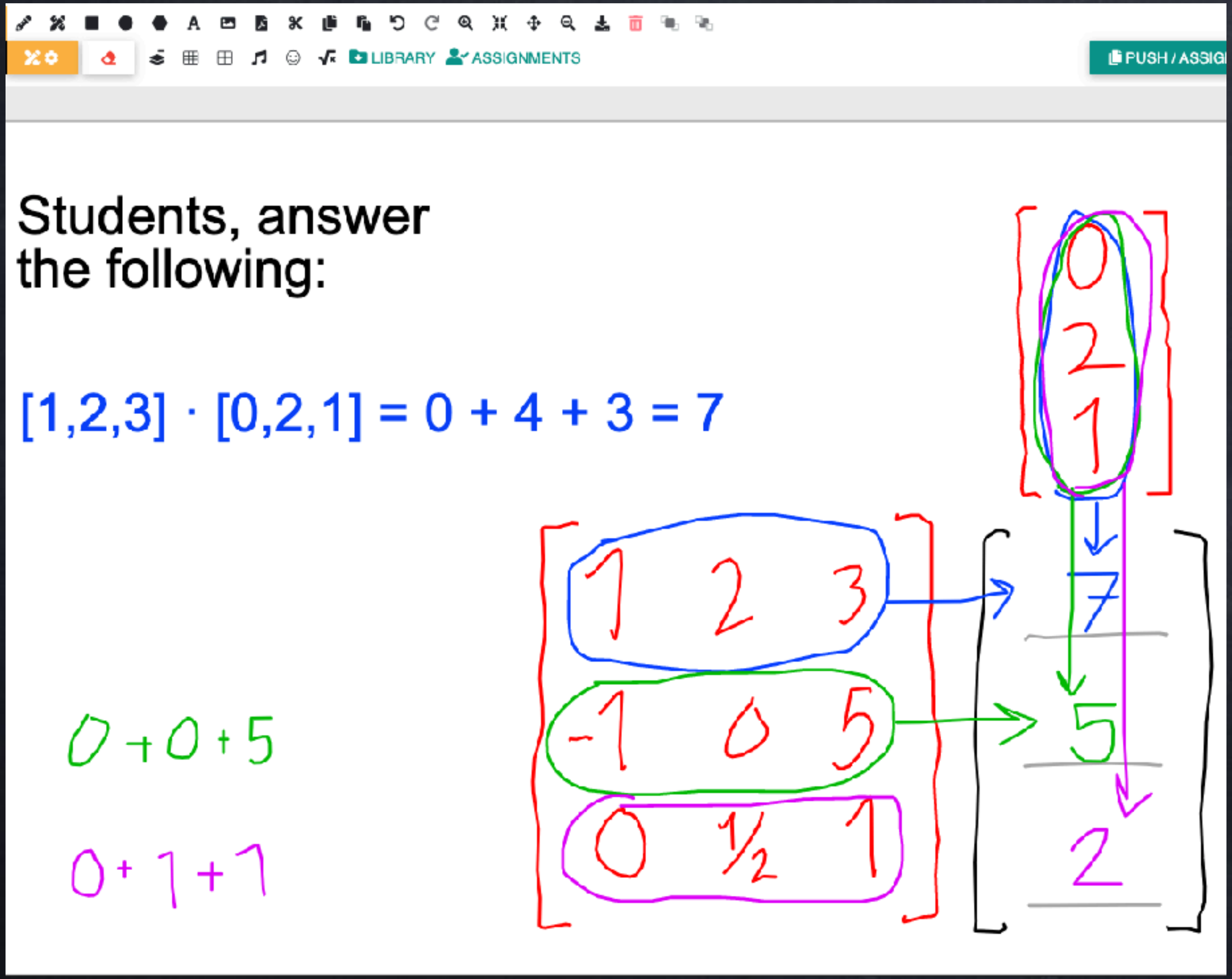
Can be done by computers:

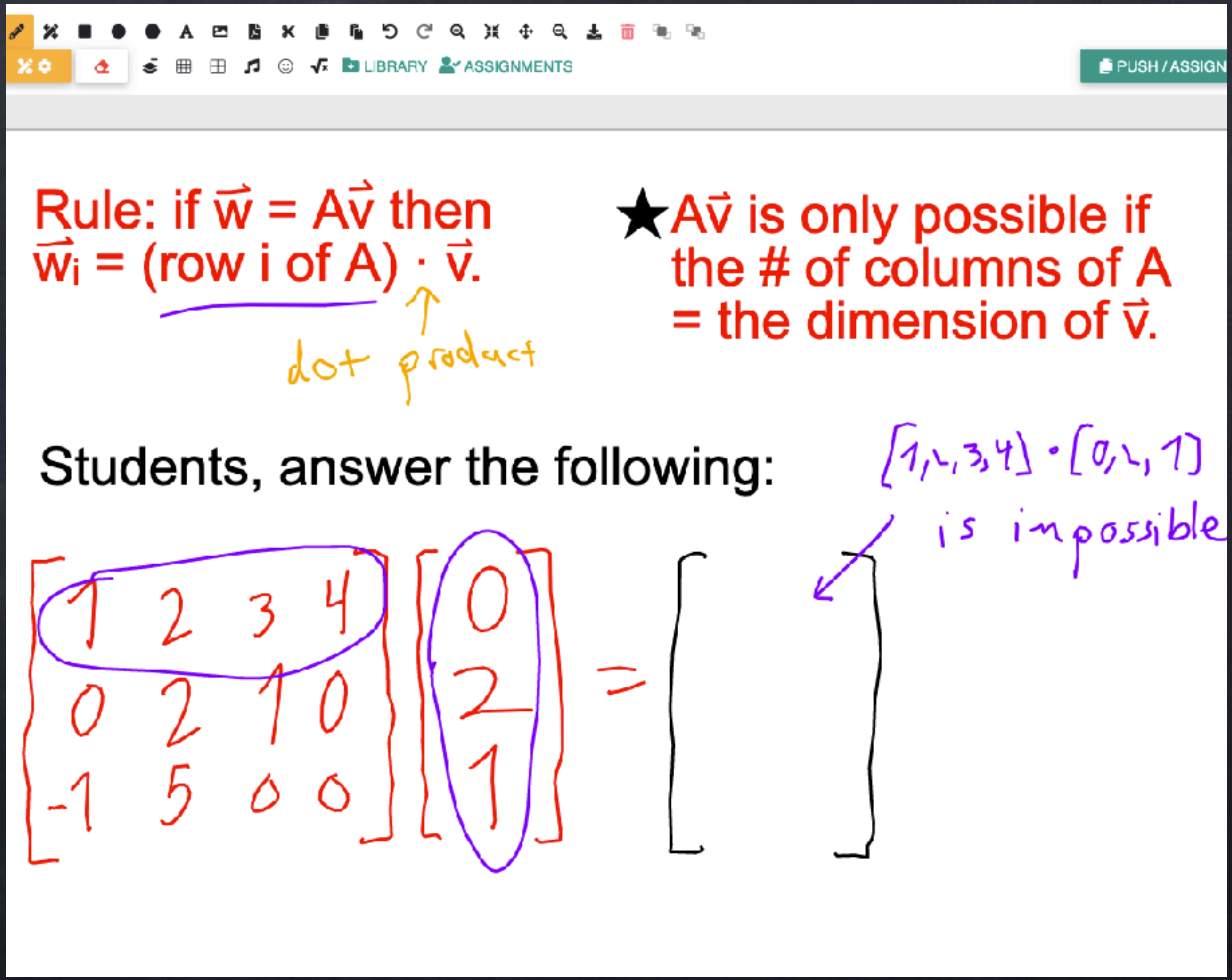
- Find the distance between $(4, 0, 7)$ and $(1, -9, 9)$.
- Calculate $[1,2] \cdot [3,4]$.
- Find the angle between the vectors $\vec{u} = [4, 2]$ and $\vec{w} = [-4, 6]$.
- Multiply $\begin{bmatrix} -2 & 8 & 3 \\ 11 & 0 & 9 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ -6 \end{bmatrix}$.

Cannot:

- Is $[6, 1]$ a scalar multiple of $[18, 3]$?
- If $|\vec{a}| = 8$ and $|\vec{b}| = 2$ and $\vec{a} \cdot \vec{b} = -16$, find the angle between $\vec{a}$ and $\vec{b}$.
- Are the planes $x + y - z = 2$ and $2x + 2y - 4z = 7$ parallel?
- If A is a 3×5 matrix and B is 5×4, what are the dimensions of AB?

Previously: matrix multiplication





Previously: matrix multiplication

Rule: if $M = AB$ then m_{ij} = (row i of A) · (column j of B).

Example 1: again $\begin{bmatrix} 4 & 1 & 0 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 9 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 15 & 16 \end{bmatrix}$

2×3 $\quad$ 3×2 $\quad$ 2×2

The "inner" numbers must agree for AB to exist.

The "outer" numbers give the dimensions of AB.

Rule: if $M = AB$ then m_{ij} = (row i of A) · (column j of B).

Example 3: $\begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 11 & 8 \\ 1 & 9 \end{bmatrix}$

Example 4: $\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 7 & 17 \\ 0 & 13 \end{bmatrix}$

★ When multiplying matrices, AB and BA can be different!

Bad math

- We will *never* combine $[a, b]$ and $[c, d]$ into $[ac, bd]$.

  We will *never* combine $\begin{bmatrix} a \\ b \end{bmatrix}$ and $\begin{bmatrix} c \\ d \end{bmatrix}$ into $\begin{bmatrix} ac \\ bd \end{bmatrix}$.

  We will *never* combine $[a, b]$ and $\begin{bmatrix} c \\ d \end{bmatrix}$ into $\begin{bmatrix} ac \\ bd \end{bmatrix}$.

  - There is the dot product: $[a, b] \cdot [c, d] = ac + bd$.

    The dot product of two vectors is a *number*.

    It only exists if the vectors have the same dimension.

- We will *never* combine $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\begin{bmatrix} e & f \\ g & h \end{bmatrix}$ into $\begin{bmatrix} ae & bf \\ cg & dh \end{bmatrix}$.

Bad math

- We will *never* combine $[a, b]$ and $[c, d]$ into $[ac, bd]$.
  - There is the dot product: $[a, b] \cdot [c, d] = ac + bd$.
    The dot product of two vectors is a *number*.
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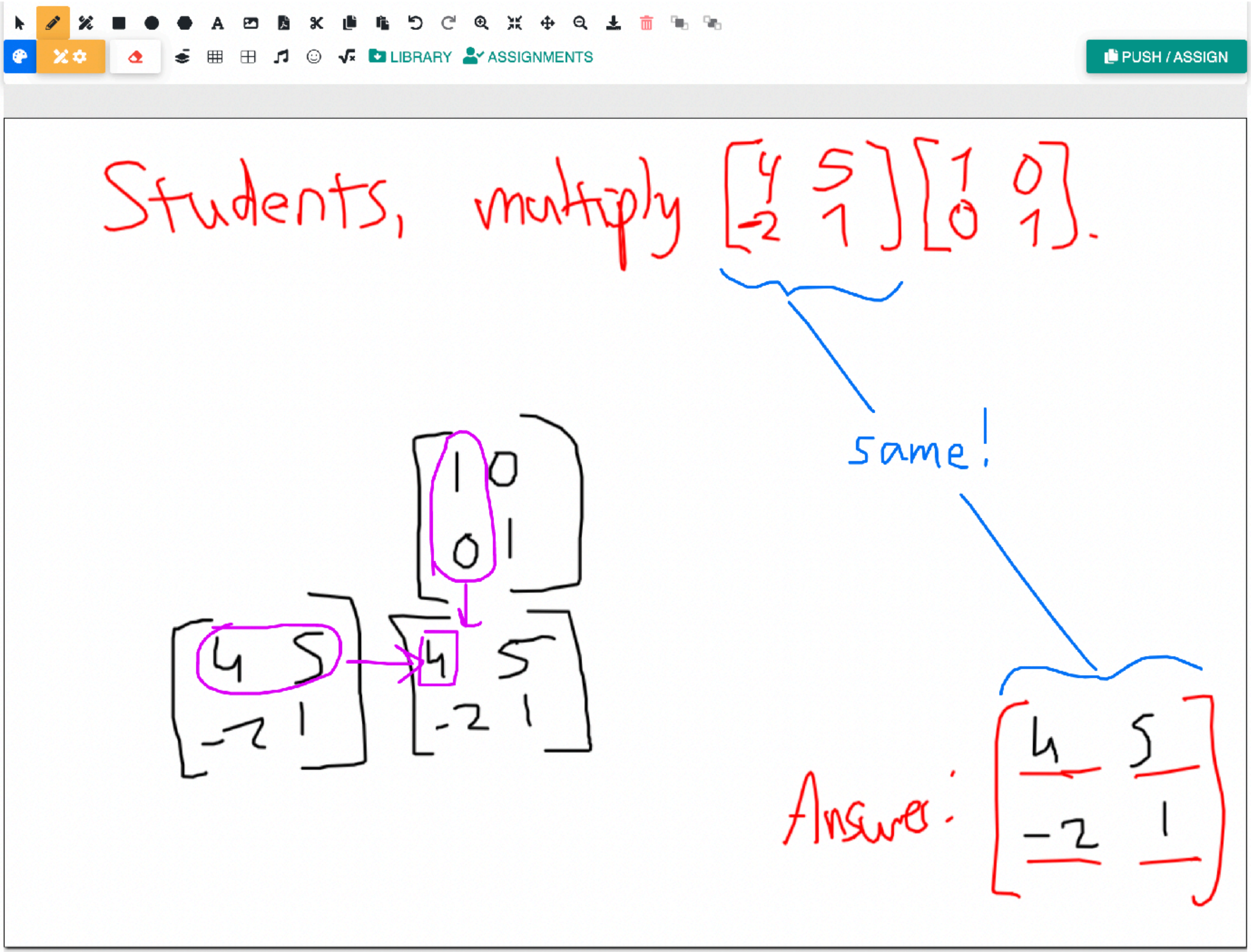
- We will *never* combine $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\begin{bmatrix} e & f \\ g & h \end{bmatrix}$ into $\begin{bmatrix} ae & bf \\ cg & dh \end{bmatrix}$.
  - There is matrix multiplication: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$

Whiteboard website

http://whiteboard.fi/math1688

- If you are asked for a room code, it is **math1688**
- Type your first and last name, then click Join Whiteboard Class
- Click TOGGLE TEACHER WHITEBOARD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Students, multiply $\begin{bmatrix} 4 & 5 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
same!
Answer: $\begin{bmatrix} 4 & 5 \\ -2 & 1 \end{bmatrix}$

Identity and inverse

A **square matrix** has the same numbers of rows as columns.

- If we use only $n \times n$ matrices, the matrix product AB always exists.

The $n \times n$ **identity matrix**, written $I_{n\times n}$ or I_n just I, is a special matrix such that

$$AI = A \quad \text{and} \quad IA = A$$

for all $n \times n$ matrices A.

- $I_{2\times2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. $I_{3\times3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. $I_{4\times4} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.
- In a way, $I_{n\times n}$ acts like the number 1.

Identity and inverse

With numbers, we know $1x = x$ and we know $x \cdot x^{-1} = x \cdot \frac{1}{x} = 1$.

The **inverse** of matrix A, which we write as A^{-1}, is a matrix that satisfies

$$AA^{-1} = I.$$

- Some matrices don't have inverses (similar to 0 for numbers).
  - We call such a matrix **non-invertible**.
  - A matrix that is not square cannot have an inverse.
- If A^{-1} does exist, then it is unique and $A^{-1}A = I$ is also true.

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Students, multiply $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$.

$$\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

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Answer: $\begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

The "2 by 2 identity matrix" is $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\underbrace{\begin{bmatrix} 4/{-2} & -2/{-2} \\ -3/{-2} & 1/{-2} \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This means ⤴ is the "inverse" of $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

Identity and inverse

For 2×2 matrices there is a simple formula:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

- The number $ad - bc$ is called the **determinant** of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.
  - The inverse A^{-1} exists if and only if the determinant of A is *not* 0.
  - We write $\det(A)$ for the determinant of matrix A.
    Some textbooks write $|A|$, but determinants can be < 0.

For larger square matrices, $\begin{bmatrix} a & \cdots \\ \vdots & \ddots \end{bmatrix}^{-1} = \frac{1}{\det(A)} \begin{bmatrix} ?? & \cdots \\ \vdots & \ddots \end{bmatrix}$ in some way.

Previously: systems of eqns

Systems of linear equations

Examples:

$$\begin{cases} 3x - 7y = 4 \\ x + 8y = 2 \end{cases}$$

$$\begin{cases} 7x + 2y + 9z = -9 \\ 7x + 2y + 9z = 4 \\ -4x - 3y + 5z = 2 \end{cases}$$

$$\begin{cases} 7a + 2b + 9c = -9 \\ 8a \qquad + 6c = 0 \\ -4a - 3b + 5c = 2 \end{cases}$$

$$\begin{cases} 3x - 7y = 4 \\ x + 8y = 2 \\ 2x + 5y = 7 \end{cases}$$

$$\begin{cases} 7x + 2y + 9z = -9 \\ -4x - 3y + kz = 2 \end{cases}$$

$$\begin{cases} 6x_1 + 2x_2 - 5x_3 + x_4 = 1 \\ 5x_1 \qquad - 7x_3 + 2x_4 = 3 \end{cases}$$

There are many methods to solve systems of linear equations by hand.

- Substitution
- Elimination

} Fewer calculations, but you have to be clever about what steps to take.

- Matrix inverse*
- Cramer's Rule*

} Follow the same steps every time, but do a lot of calculations.

It is also possible to determine the number of solutions—zero, one, or infinity—without actually solving the system.

- Determinant*
- Rank

* only when # of equations = # of variables

$$4x - 2y = 14$$
$$2x + y = 15$$

can be written as one matrix eqn:

$$\begin{bmatrix} 4 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 14 \\ 15 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -2 \\ 2 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 4 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 2 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 14 \\ 15 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1/8 & 1/4 \\ -1/4 & 1/2 \end{bmatrix}\begin{bmatrix} 4 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}}_{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}\text{, which is }\begin{bmatrix} x \\ y \end{bmatrix}} = \begin{bmatrix} 1/8 & 1/4 \\ -1/4 & 1/2 \end{bmatrix}\begin{bmatrix} 14 \\ 15 \end{bmatrix} =$$

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Students, compute $\begin{bmatrix} 1/8 & 1/4 \\ -1/4 & 1/2 \end{bmatrix} \begin{bmatrix} 14 \\ 15 \end{bmatrix}$

$$= \begin{bmatrix} 7/4 + 15/4 \\ -7/2 + 15/2 \end{bmatrix} = \begin{bmatrix} 22/4 \\ 8/2 \end{bmatrix} = \begin{bmatrix} 5.5 \\ 4 \end{bmatrix}$$

Answer: $\begin{bmatrix} 5.5 \\ 4 \end{bmatrix}$

$$4x - 2y = 14$$
$$2x + y = 15$$

↓

$$x = \frac{11}{2}$$
$$y = 4$$

Solving systems using matrices

A system of n linear equations with m variables can be written as a single matrix equation

$$AX = B \quad \text{or} \quad A\overrightarrow{x} = \overrightarrow{b},$$

where we can think of X and B as either $m \times 1$ matrices or as vectors. The $n \times m$ matrix A is called the "coefficient matrix".

- If $n = m$, then A is a square matrix, so it has a determinant $\det(A)$.
  - If $\det(A) \neq 0$, then the solution to the system is exactly

$$X = A^{-1}B.$$