Math 1688

Thursday, 13 January

Warm-up:
Matrix multiplication.

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$$\begin{bmatrix} 14 & -21 & -4 \\ -31 & 47 & 9 \\ 7/2 & -11/2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 2 & -2 \\ 1 & 0 & -4 \\ 12 & 7 & 14 \end{bmatrix} \begin{bmatrix} 1 & & * \\ & * & \\ * & * & * \end{bmatrix}$$

$$\begin{bmatrix} 14 & -21 & -4 \\ -31 & 47 & 9 \\ {}^{7}/_{2} & {}^{-11}/_{2} & -1 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 2 & -2 \\ 1 & 0 & -4 \\ 12 & 7 & 14 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Last time: identity and inverse

The **$n \times n$ identity matrix**, written $I_{n\times n}$ or I_n just I, is

$$I_{2\times2} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}. \quad I_{3\times3} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}. \quad I_{4\times4} = \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}.$$

- Like "$1 \times n = n$" for numbers, the matrix I satisfies $IM = M$ for any M.

The **inverse** of the matrix A, written A^{-1}, is the matrix for which $A^{-1}A = I$.

- This is like $\frac{3}{2}$ for the number $\frac{2}{3}$ because $\frac{3}{2} \times \frac{2}{3} = 1$.
- Some matrices do not have an inverse (for numbers, 0 has no inverse).

The **determinant** of a square matrix M is a number written $\det(M)$.

- If $\det(M) = 0$ then M^{-1} does not exist.

Last time: identity and inverse

For 2×2 matrices there are simple formulas:

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

and

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

For larger square matrices, $\begin{bmatrix} a & \cdots \\ \vdots & \ddots \end{bmatrix}^{-1} = \frac{1}{\det(A)}\begin{bmatrix} ?? & \cdots \\ \vdots & \ddots \end{bmatrix}$ in some way.

3×3 identity matrix

$$\begin{bmatrix} 5 & 2 & -2 \\ 1 & 0 & -4 \\ 12 & 7 & 14 \end{bmatrix} \begin{bmatrix} 14 & -21 & -4 \\ -31 & 47 & 9 \\ 7/2 & -11/2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 2 & -2 \\ 1 & 0 & -4 \\ 12 & 7 & 14 \end{bmatrix}^{-1} = \begin{bmatrix} 14 & -21 & -4 \\ -31 & 47 & 9 \\ 7/2 & -11/2 & -1 \end{bmatrix}$$

The inverse of $\begin{bmatrix} 5 & 2 & -2 \\ 1 & 0 & -4 \\ 12 & 7 & 14 \end{bmatrix}$ is $\begin{bmatrix} 14 & -21 & -4 \\ -31 & 47 & 9 \\ 7/2 & -11/2 & -1 \end{bmatrix}$.

Applications of matrices

Matrices (the plural of “matrix”) can be used for

- *systems of equations*
- geometry / linear transformations
- network/graph analysis
- probability and statistics
- cryptography
- image compression
- physics - optics, electronics, quantum

and more.

Solving systems using matrices

The system of three equations

$$\begin{cases} 5x + 2y - 2z = 4 \\ x - 4z = 2 \\ 12x + 7y + 14z = 5 \end{cases}$$

can be written as the single equation

$$\begin{bmatrix} 5 & 2 & -2 \\ 1 & 0 & -4 \\ 12 & 7 & 14 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix}$$

using matrices.

We usually write this as $AX = B$ and call A the **matrix of coefficients**.

Solving systems using matrices

$$AX = B$$

(A is n×n matrix)

$$A^{-1}AX = A^{-1}B$$

recall $A^{-1}A = I$

$$IX = A^{-1}B$$

recall $IX = X$

$$X = A^{-1}B$$

where I is the n × n identity matrix ($\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ for 2×2, $\begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$ for larger).

Let's also think about

$$3x = 5$$

$$\frac{1}{3} \cdot 3x = \frac{1}{3} \cdot 5$$

$$1x = \frac{1}{3} \cdot 5$$

$$x = \frac{1}{3} \cdot 5$$

Solving systems using matrices

Any system of linear eqns corresponds to a single equation $AX = B$. Example:

$$\begin{cases} 5x + 2y - 2z = 4 \\ x - 4z = 2 \\ 12x + 7y + 14z = 5 \end{cases} \to \begin{bmatrix} 5 & 2 & -2 \\ 1 & 0 & -4 \\ 12 & 7 & 14 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix}.$$

If the coefficient matrix A is invertible, then we can solve the system as

$$X = A^{-1}B.$$

Example:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 & 2 & -2 \\ 1 & 0 & -4 \\ 12 & 7 & 14 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 14 & -21 & -4 \\ -31 & 47 & 9 \\ \frac{7}{2} & \frac{-11}{2} & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} -6 \\ 15 \\ -2 \end{bmatrix}.$$

Eqn with square matrix

- We can solve the matrix equation $AX = B$ as just
$$X = A^{-1}B$$
if we first compute the inverse of the matrix A.

- There is also a direct formula for each variable: writing "A_i" for the matrix formed by replacing $\text{Column } i$ of matrix A with the single column B, **Cramer's Rule** says that
$$x_i = \frac{\det(A_i)}{\det(A)}.$$

Eqn with square matrix

Example: solve $\begin{cases} 5x - 2y = 15 \\ x + 4y = 14 \end{cases}$ using an inverse matrix.

$$\begin{bmatrix} 5 & -2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 15 \\ 14 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ 1 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 15 \\ 14 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{4}{22} & \frac{2}{22} \\ \frac{-1}{22} & \frac{5}{22} \end{bmatrix} \begin{bmatrix} 15 \\ 14 \end{bmatrix} = \begin{bmatrix} \frac{4}{22}(15) + \frac{2}{22}(14) \\ \frac{-1}{22}(15) + \frac{5}{22}(14) \end{bmatrix} = \boxed{\begin{bmatrix} 4 \\ 5/2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}}$$

Eqn with square matrix

Example: solve $\begin{cases} 5x - 2y = 15 \\ x + 4y = 14 \end{cases}$ using Cramer's Rule.

$$x = \frac{\det\left(\begin{bmatrix} 15 & -2 \\ 14 & 4 \end{bmatrix}\right)}{\det\left(\begin{bmatrix} 5 & -2 \\ 1 & 4 \end{bmatrix}\right)} = \frac{15(4) - (-2)14}{5(4) - (-2)1} = \frac{88}{22} = 4 = x$$

$$y = \frac{\det\left(\begin{bmatrix} 5 & 15 \\ 1 & 14 \end{bmatrix}\right)}{\det\left(\begin{bmatrix} 5 & -2 \\ 1 & 4 \end{bmatrix}\right)} = \frac{5(14) - (15)1}{5(4) - (-2)1} = \frac{55}{22} = \frac{5}{2} = y$$

Problems with X = $A^{-1}B$ & Cramer

This method only possible if

- A has the same number of rows as columns (a "square" matrix)
- and $\det(A) \neq 0$.

Otherwise, A^{-1} does not exist.

If $\det(A) = 0$, the system may or may not have solutions.

- $\det\left(\begin{bmatrix} 6 & 3 \\ 2 & 1 \end{bmatrix}\right) = 6(1) - 2(3) = 0.$
- $\begin{cases} 6x + 3y = 10 \\ 2x + \quad y = 5 \end{cases}$ has solutions but $\begin{cases} 6x + 3y = 10 \\ 2x + \quad y = 8 \end{cases}$ does not.

Number of eqns and variables

Only square matrices have determinants. For example, the system

$$\begin{cases} 4x + 9y - z = 6 \\ \phantom{4x + {}} 2y + 3z = 0 \end{cases}$$

corresponds to the matrix equation

$$\begin{bmatrix} 4 & 9 & -1 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix},$$

but there is no determinant for this matrix.

Number of eqns and variables

A system of equations is called **consistent** if at least one solution exists.
It is called **inconsistent** if no solutions exist.

An **overdetermined** system has more equations than variables.

- Overdetermined systems are *usually* (but not always) inconsistent.

An **underdetermined** system has fewer equations than variables.

- Underdetermined systems are *usually* (but not always) consistent.

Number of eqns and variables

Overdetermined

$$\begin{cases} 3x - 4y + 2z = 8 \\ x + 7y = 2 \\ 6x + y - 3z = 1 \\ 8y + z = 0 \\ 4x + 2z = 3 \end{cases}$$

$$\begin{cases} 4x - 3y = 1 \\ x + 5y = 6 \\ 2x + y = 3 \end{cases}$$

Underdetermined

$$\begin{cases} 3x - 4y + 2z = 8 \\ 3x - 4y + 2z = 1 \end{cases}$$

$$\begin{cases} 3x - 4y + 2z = 8 \\ x + y = 2 \end{cases}$$

Related topics

Linear combinations of vectors

Linear independent* sets of vectors

Systems of linear equations: the set of solutions can be

- nothing
- a single point
- a line
- a plane
- a “hyperplane” (if you have 4 or more variables)

The rank* of the coefficient matrix helps determine which.

* We will define these later.

Linear combinations

A **linear combination** of some vectors is any sum of scalar multiples of those vectors.

- In symbols, $\overrightarrow{u}$ is a linear combination of $\overrightarrow{v}$ and $\overrightarrow{w}$ if
$$\overrightarrow{u} = s\overrightarrow{v} + t\overrightarrow{w}$$
for some numbers s, t.

- For more vectors, $\overrightarrow{u}$ is a linear combination of $\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots, \overrightarrow{v_n}$ if
$$\overrightarrow{u} = s_1\overrightarrow{v_1} + s_2\overrightarrow{v_2} + \cdots + s_n\overrightarrow{v_n}$$
for some numbers (scalars) $s_1, \ldots, s_n$.

Linear combinations

A **linear combination** of some vectors is any sum of scalar multiples of those vectors.

- In symbols, $\overrightarrow{u}$ is a linear combination of $\overrightarrow{v}$ and $\overrightarrow{w}$ if
$$\overrightarrow{u} = s\overrightarrow{v} + t\overrightarrow{w}$$
for some numbers s, t.

Example 1: Write $\begin{bmatrix} 5 \\ 24 \end{bmatrix}$ as a linear combination of $\overrightarrow{v_1} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$ and $\overrightarrow{v_2} = \begin{bmatrix} 3 \\ -9 \end{bmatrix}$.

Example 2: $\begin{bmatrix} 5 \\ 24 \end{bmatrix}$ *cannot* be written as a linear combination of $\overrightarrow{v_1} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ and $\overrightarrow{v_2} = \begin{bmatrix} 10 \\ 2 \end{bmatrix}$. Why?

$\overrightarrow{v_1}$ and $\overrightarrow{v_2}$ point in the same direction, so any scalar multiples of them will too, and so <u>all</u> sums of s.m. of $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$ are parallel to <5,1>. Since <5,24> is <u>not</u> parallel to <5,1>, it cannot be a sum of scalar multiples of $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$.

Linear dependence

Equivalent definitions:

- A *set of vectors* is called **linearly dependent** if one vector is a linear combination of the others.

- A set $\{\overrightarrow{v_1}, \ldots, \overrightarrow{v_n}\}$ is **linearly dependent** if there exist numbers $s_1, \ldots, s_n$ not *all* zero such that

$$s_1\overrightarrow{v_1} + s_2\overrightarrow{v_2} + \cdots + s_n\overrightarrow{v_n} = \overrightarrow{0}.$$

  - Note: *some* of the s_i can be zero, just not all.

- A set is **linearly dependent** if it is not linearly independent.

Linear independence

Equivalent definitions:

- A *set of vectors* is called **linearly independent** if no vector is a linear combination of the others.

- A set $\{\overrightarrow{v_1}, \ldots, \overrightarrow{v_n}\}$ is **linearly independent** if the *only* solution to

$$s_1\overrightarrow{v_1} + s_2\overrightarrow{v_2} + \cdots + s_n\overrightarrow{v_n} = \overrightarrow{0}$$

  is $s_1 = s_2 = \cdots = s_n = 0$.

- A set is **linearly independent** if it is not linearly dependent.

Linear (in)dependence

- Linearly dependent: one vector is a linear combination of the others.
  - Example: $\left\{ \begin{bmatrix} 5 \\ 24 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is linearly dependent.
- Linearly independent: no vector is a linear combination of the others.
  - Example: $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is linearly independent.
- Note that a single vector isn't called linearly dependent or independent. This is about sets of vectors.

Linear (in)dependence

- Linearly dependent: one vector is a linear combination of the others.
  - Example: $\begin{bmatrix} 5 \\ 24 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are linearly dependent.
- Linearly independent: <u>no</u> vector is a linear combination of the others.
  - Example: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are linearly independent.
- Note that a single vector isn't called linearly dependent or independent. This is about sets of vectors.

Linear (in)dependence

- Example: Determine whether

$$\left\{ \begin{bmatrix} -1 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 12 \end{bmatrix}, \begin{bmatrix} -1 \\ 14 \\ 27 \end{bmatrix} \right\}$$

is linearly dependent or linearly independent.

We should try to find numbers x and y for which

$$x \begin{bmatrix} -1 \\ 5 \\ 7 \end{bmatrix} + y \begin{bmatrix} 4 \\ -2 \\ 12 \end{bmatrix} = \begin{bmatrix} -1 \\ 14 \\ 27 \end{bmatrix}.$$

Indeed, we can: $x = 3$ and $y = 1/2$.

So these vectors are linearly dependent.

Linear (in)dependence

- Some facts to notice:

> If a set contains the zero vector then it is linearly dependent.
>
> If the vectors are d-dimensional (each is a list of d numbers), then any set of $d+1$ or more vectors will be linearly dependent.

- Examples:
  - $\left\{ \begin{bmatrix} 1 \\ -9 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 7 \end{bmatrix} \right\}$ must be LD. Note $0\overrightarrow{v_1} + 5\overrightarrow{v_2} + 0\overrightarrow{v_3} = \overrightarrow{0}$.
  - $\left\{ \begin{bmatrix} 3 \\ -8 \end{bmatrix}, \begin{bmatrix} 5 \\ 9 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ must be LD.

Rank

The **rank** of a matrix is the number of linearly independent columns.

An $n \times m$ matrix can have rank at most $\min(n, m)$.
An $n \times m$ matrix is called **full rank** if its rank is equal to $\min(n, m)$.