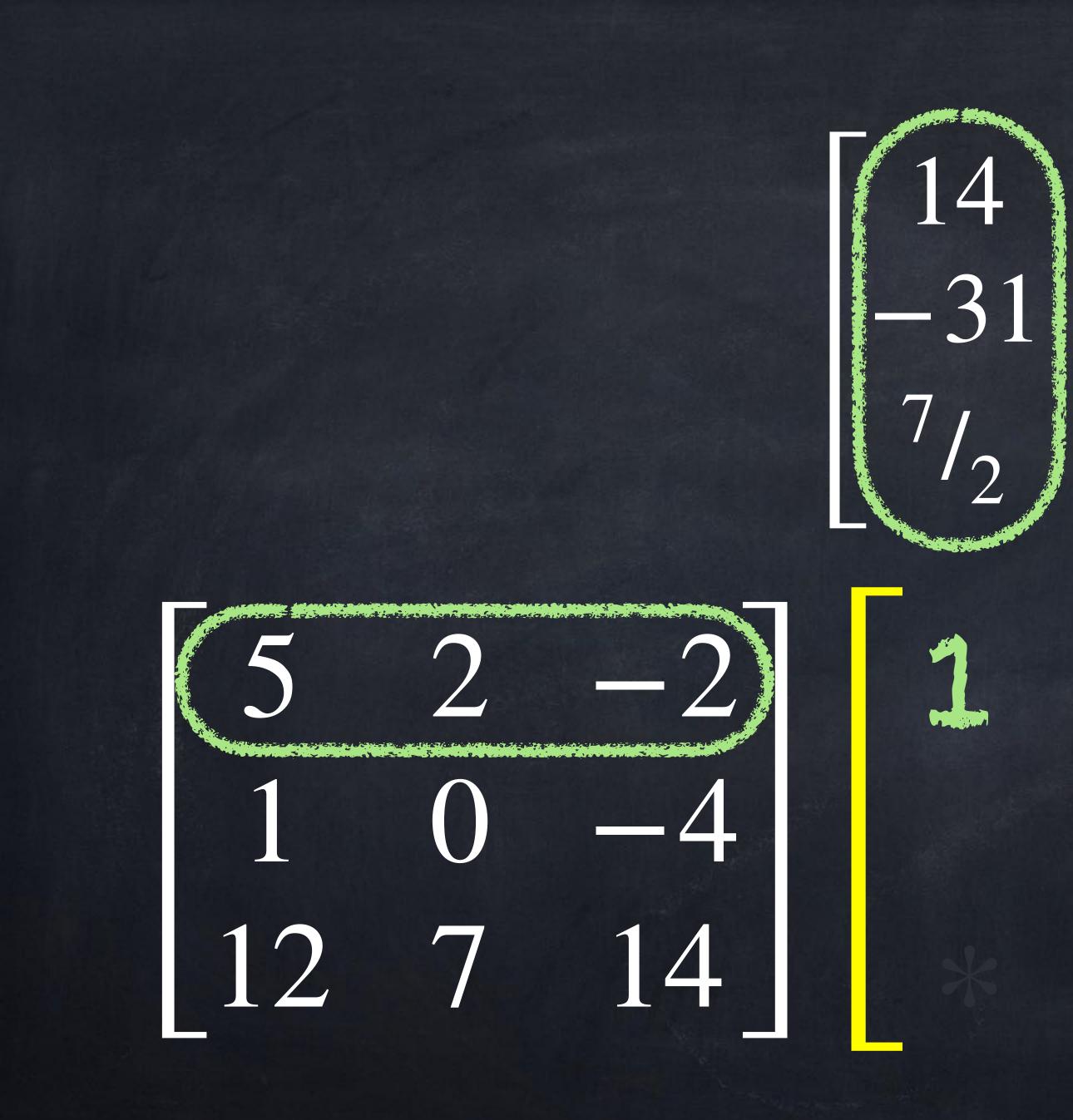


Warm-up: Matrix multiplication.

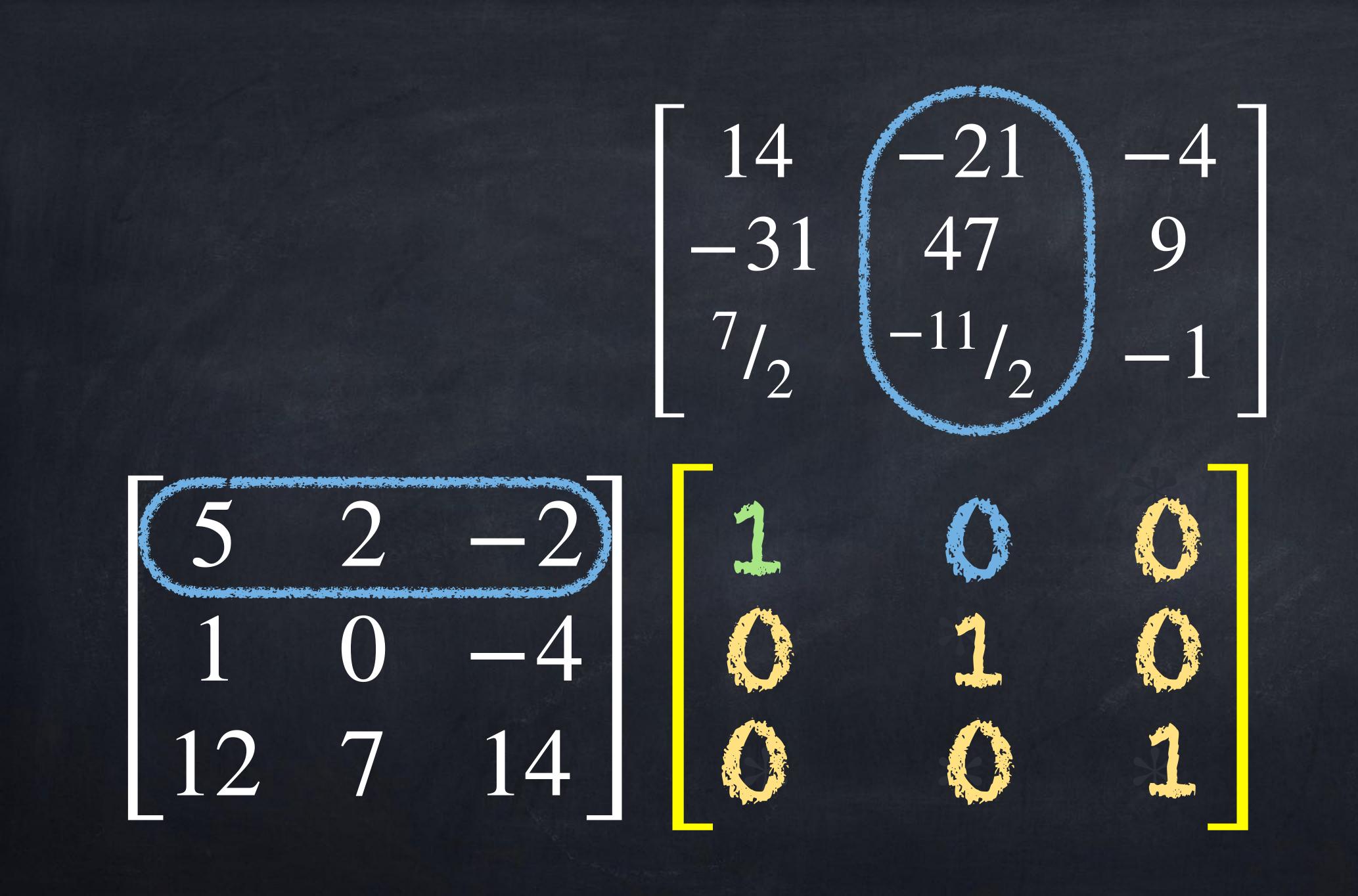
theadamabrams.com/live

### Thursday, 13 January





 $\begin{array}{c} 14 & -21 \\ -31 & 47 \end{array}$ -4 9 -11/ \_\_1 17



## Last time: identity and inverse The $n \times n$ identity matrix, written $I_{n \times n}$ or $I_n$ just I, is $I_{2\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} . \quad I_{3\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \quad I_{4\times 4} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} .$

• This is like  $\frac{3}{2}$  for the number  $\frac{2}{3}$  because  $\frac{3}{2} \times \frac{2}{3} = 1$ .

The determinant of a square matrix M is a number written det(M). • If det(M) = 0 then  $M^{-1}$  does not exist.

• Like " $1 \times n = n$ " for numbers, the matrix I satisfies M = M for any M.

The inverse of the matrix A, written  $A^{-1}$ , is the matrix for which  $A^{-1}A = I$ . Some matrices do not have an inverse (for numbers, 0 has no inverse).



### For $2 \times 2$ matrices there are simple formulas:

 $det\left( \begin{vmatrix} a & b \\ c & d \end{vmatrix} \right) =$ 

### For larger square matrices,



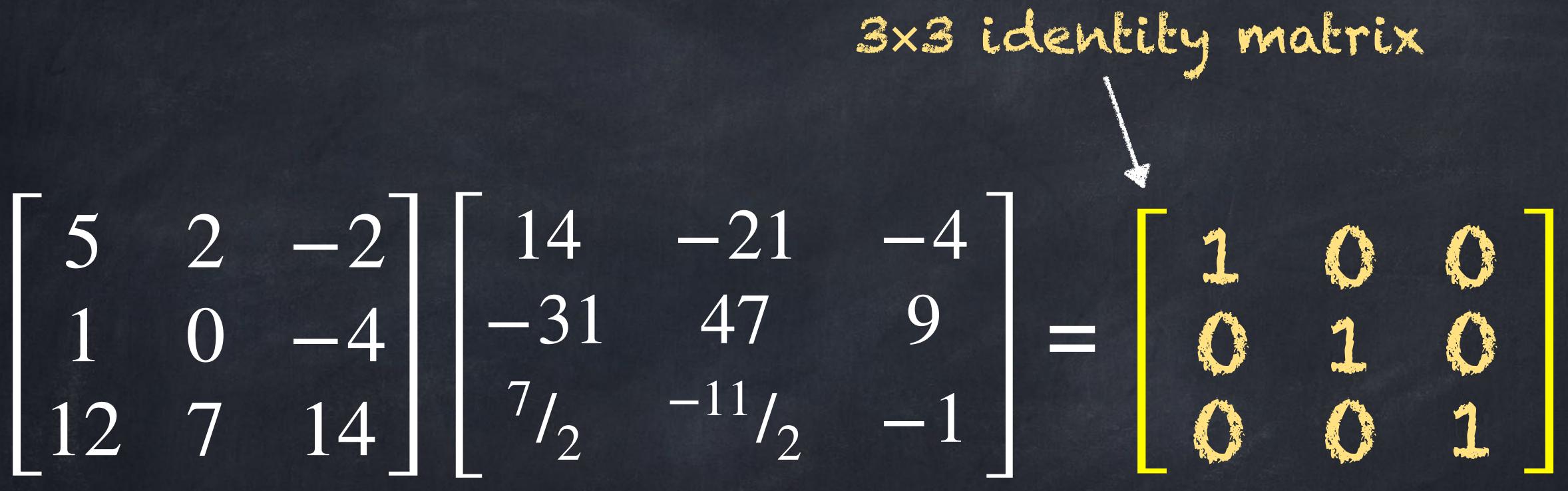
$$= ad - bc$$

### and

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\det(A)} \begin{bmatrix} ?? & \cdots \\ \vdots & \ddots \end{bmatrix}$$
 in some way.







 $\begin{bmatrix} 5 & 2 & -2 \\ 1 & 0 & -4 \\ 12 & 7 & 14 \end{bmatrix}^{-1} = \begin{bmatrix} 14 & -21 & -4 \\ -31 & 47 & 9 \\ 7/_2 & ^{-11}/_2 & -1 \end{bmatrix}$ 

The inverse of  $\begin{bmatrix} 5 & 2 & -2 \\ 1 & 0 & -4 \\ 12 & 7 & 14 \end{bmatrix}$  is  $\begin{bmatrix} 14 & -21 & -4 \\ -31 & 47 & 9 \\ 7/_2 & -11/_2 & -1 \end{bmatrix}$ .

## Matrices (the plural of "matrix") can be used for

- systems of equations
- geometry / linear transformations 0
- network/graph analysis
- probability and statistics
- cryptography 0
- image compression 0
- physics optics, electronics, quantum 0 and more.





### The system of three equations

 $\begin{cases} 5x + 2y - 2z = 4\\ x - 4z = 2\\ 12x + 7y + 14z = 5 \end{cases}$ can be written as the single equation using matrices. We usually write this as AX = B and call A the matrix of coefficients.







AX = BA - 1AX = A - 1Brecall  $A^{-1}A = I$ IX = A-113 recall IX = X X = A-113

where I is the n x n identity matrix  $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$  for 2×2,  $\begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$ for larger).

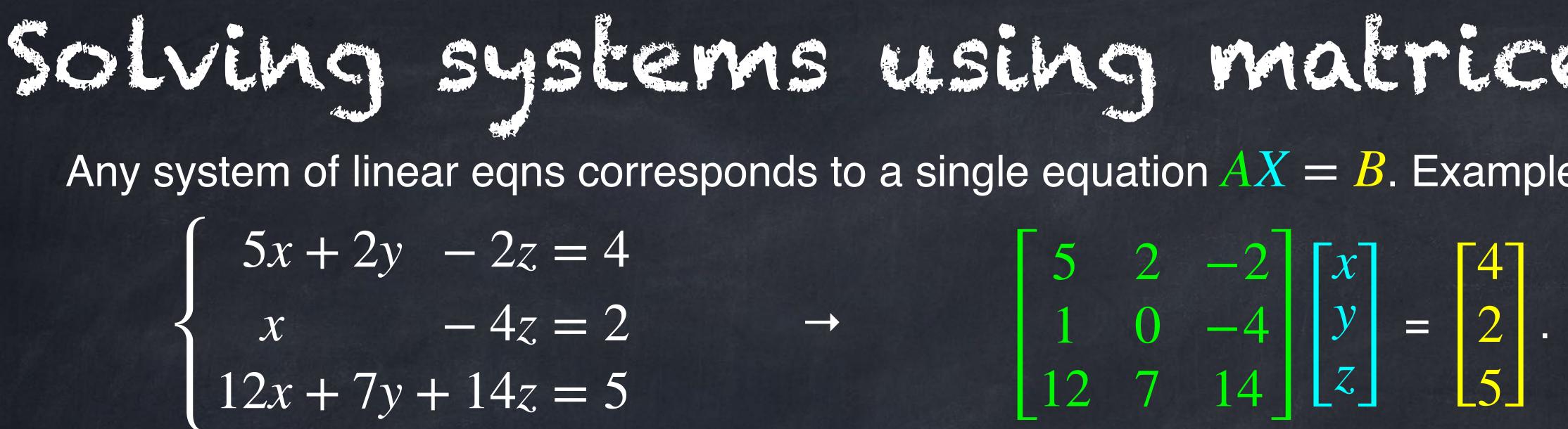
## Solving systems using matrices

(A is nxn matrix)

### Let's also think about 3x = 5

 $1 \times = \frac{1}{2} \le$ 





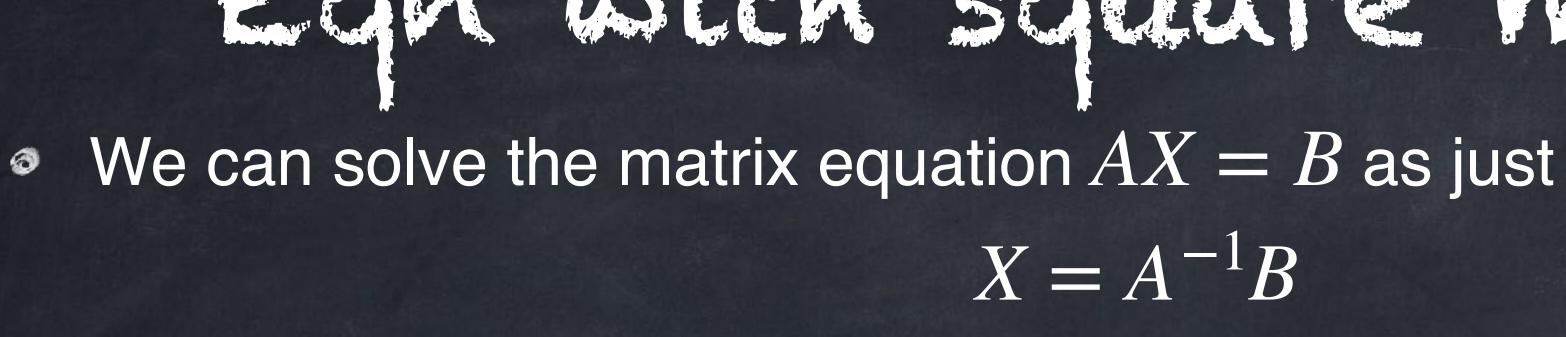
If the coefficient matrix A is invertible, then we can solve the system as  $X = A^{-1}B$ 

Example:

## Solving systems using matrices Any system of linear equst corresponds to a single equation AX = B. Example:

 $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 & 2 & -2 \\ 1 & 0 & -4 \\ 12 & 7 & 14 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 14 & -21 & -4 \\ -31 & 47 & 9 \\ \frac{7}{2} & \frac{-11}{2} & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} -6 \\ 15 \\ -2 \end{bmatrix}.$ 





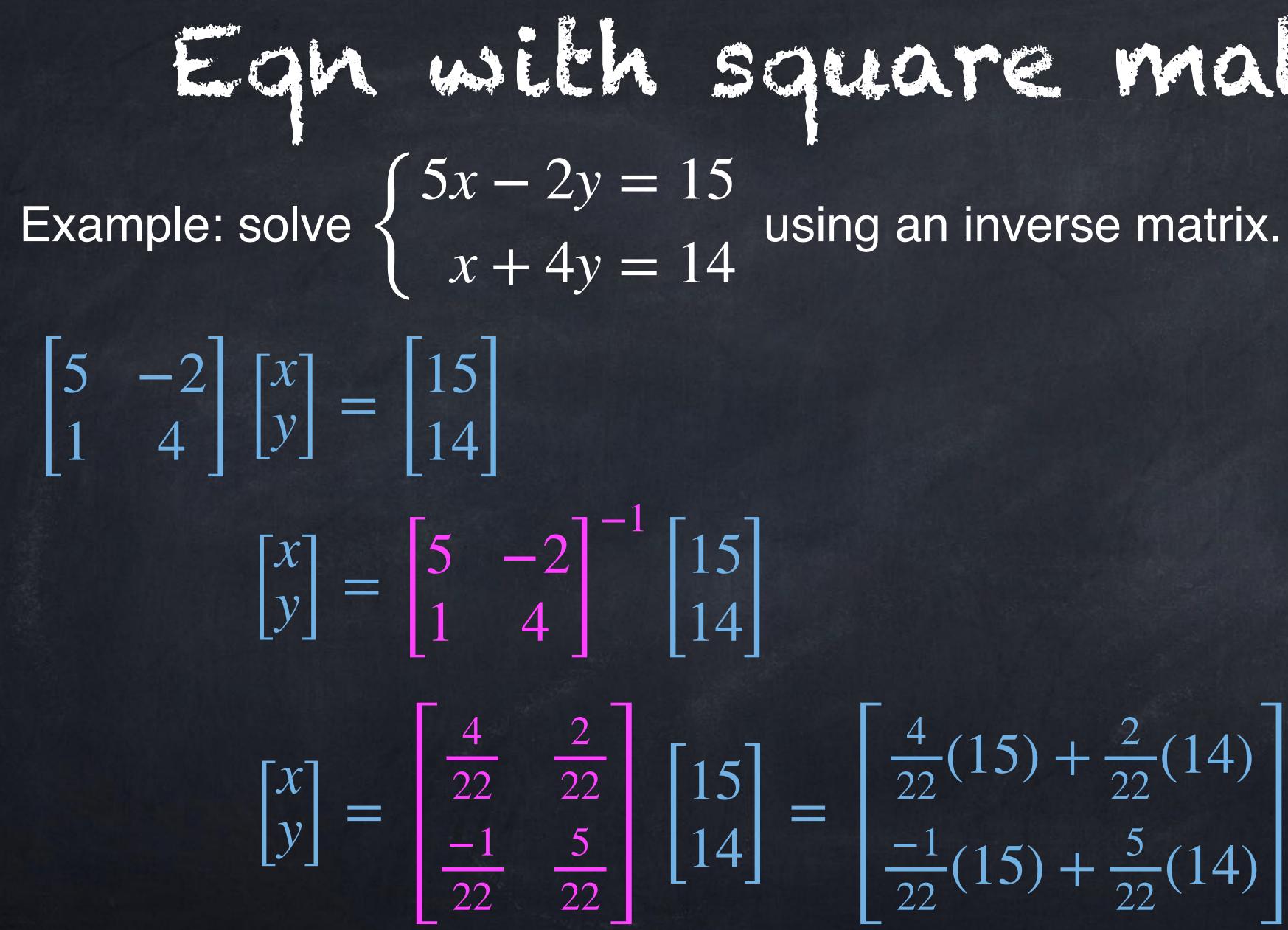
if we first compute the inverse of the matrix A.

column *B*, **Cramer's Rule** says that

Ean with square matrix  $X = A^{-1}B$ 

• There is also a direct formula for each variable: writing " $A_i$ " for the matrix formed by replacing Column *i* of matrix A with the single

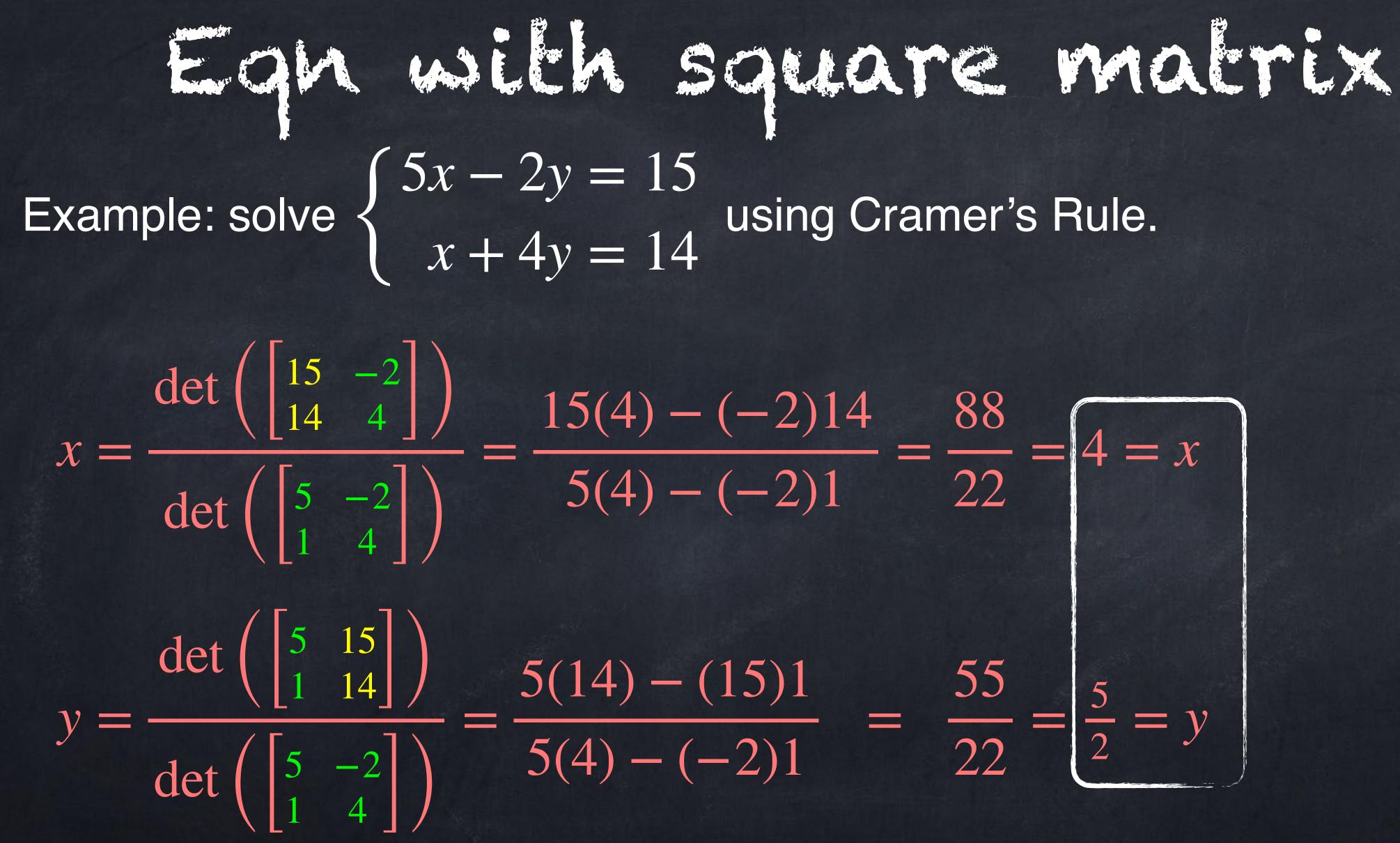
 $x_i = \frac{\det(A_i)}{\det(A)}.$ 



## Egh with square matrix

 $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{4}{22} & \frac{2}{22} \\ \frac{-1}{22} & \frac{5}{22} \end{bmatrix} \begin{bmatrix} 15 \\ 14 \end{bmatrix} = \begin{bmatrix} \frac{4}{22}(15) + \frac{2}{22}(14) \\ \frac{-1}{22}(15) + \frac{5}{22}(14) \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$ 



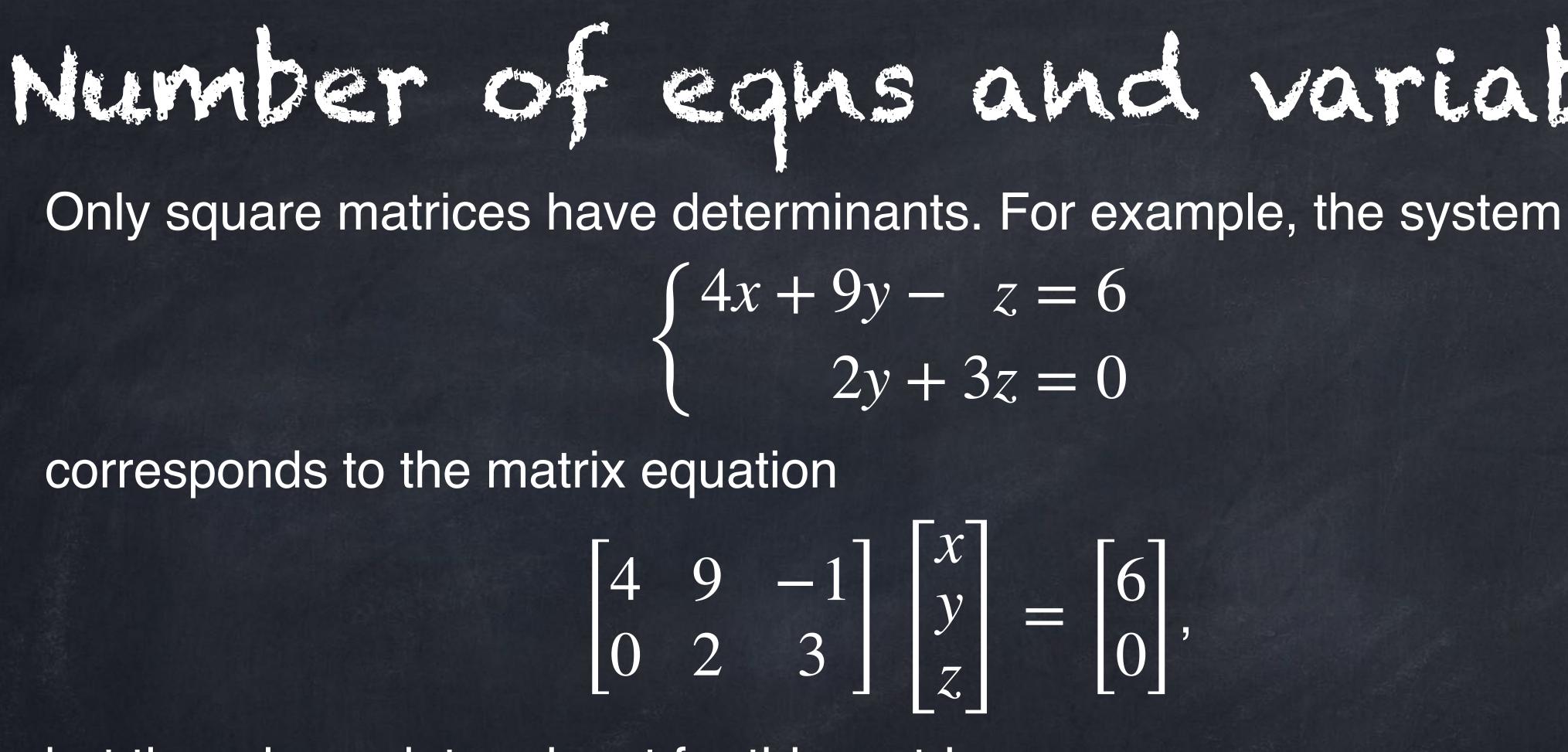




This method only possible if A has the same number of rows as columns (a "square" matrix) • and  $det(A) \neq 0$ . Otherwise,  $A^{-1}$  does not exist.

If det(A) = 0, the system may or may not have solutions. • det  $\left( \begin{bmatrix} 6 & 3 \\ 2 & 1 \end{bmatrix} \right) = 6(1) - 2(3) = 0.$  $\begin{array}{l} \bullet \quad \begin{cases} 6x + 3y = 10 \\ 2x + y = 5 \end{cases} \quad \text{has solutions but} \quad \begin{cases} 6x + 3y = 10 \\ 2x + y = 8 \end{cases} \quad \text{does not.}
\end{array}$ 





but there is no determinant for this matrix.

## Number of equs and variables

- $\begin{bmatrix} 4 & 9 & -1 \\ 0 & 2 & 3 \end{bmatrix} \begin{vmatrix} x \\ y \\ z \end{vmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix},$



## Number of equs and variables

A system of equations is called **consistent** if at least one solution exists. It is called **inconsistent** if no solutions exist.

An overdetermined system has more equations than variables.
Overdetermined systems are *usually* (but not always) inconsistent.

An underdetermined system has fewer equations than variables.
Underdetermined systems are *usually* (but not always) consistent.





### Overdetermined

	$\int 3x - 4y$	+ 2z	=	8
	x + 7y		Η	2
ł	6x + y	-3z	=	1
	8y	+ z	=	0
	4 <i>x</i>	+ 2z		3

4x - 3y = 1x + 5y = 62x + y = 3

## Number of equs and variables

Underdetermined

 $\begin{cases} 3x - 4y + 2z = 8 \\ 3x - 4y + 2z = 1 \end{cases}$ 

$$\begin{cases} 3x - 4y + 2z = 8\\ x + y = 2 \end{cases}$$





Linear combinations of vectors Linear independent\* sets of vectors

- Systems of linear equations: the set of solutions can be
- o nothing
- a single point
- a line
- a plane
- a "hyperplane" (if you have 4 or more variables) 0

The rank\* of the coefficient matrix helps determine which.

\* We will define these later.

## Lincar complinations

A linear combination of some vectors is any sum of scalar multiples of those vectors. In symbols,  $\overrightarrow{u}$  is a linear combination of  $\overrightarrow{v}$  and  $\overrightarrow{w}$  if  $\overrightarrow{u} = \overrightarrow{sv} + t\overrightarrow{w}$ 

for some numbers s, t.

• For more vectors,  $\vec{u}$  is a linear combination of  $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}$  if for some numbers (scalars)  $S_1, \ldots, S_n$ .

 $\overrightarrow{u} = s_1 \overrightarrow{v_1} + s_2 \overrightarrow{v_2} + \dots + s_n \overrightarrow{v_n}$ 

## Linear combinations

## A linear combination of some vectors.

In symbols,  $\overrightarrow{u}$  is a linear combination of  $\overrightarrow{v}$  and  $\overrightarrow{w}$  if  $\overrightarrow{u} = s \overrightarrow{v} + t \overrightarrow{w}$ 

for some numbers *s*, *t*.

# Example 1: Write $\begin{bmatrix} 5\\24 \end{bmatrix}$ as a linear combination of $\overrightarrow{v_1} = \begin{bmatrix} 5\\-2 \end{bmatrix}$ and $\overrightarrow{v_2} = \begin{bmatrix} 3\\-9 \end{bmatrix}$ .

A linear combination of some vectors is any sum of scalar multiples of



# Example 2: $\begin{bmatrix} 5\\24 \end{bmatrix}$ cannot be written as a linear combination of $\vec{v_1} = \begin{bmatrix} 5\\1 \end{bmatrix}$ and $\vec{v_2} = \begin{bmatrix} 10\\2 \end{bmatrix}$ . Why?

 $\overrightarrow{v_1}$  and  $\overrightarrow{v_2}$  point in the same direction, so any scalar multiples of them will too, and so <u>all</u> sums of s.m. of  $\overrightarrow{v_1}$  and  $\overrightarrow{v_2}$  are parallel to <5,1>. Since <5,24> is <u>not</u> parallel to <5,1>, it cannot be a sum of scalar multiples of  $\overrightarrow{v_1}$  and  $\overrightarrow{v_2}$ .



Equivalent definitions:

combination of the others.

• A set  $\{\vec{v_1}, \ldots, \vec{v_n}\}$  is **linearly dependent** if there exist numbers  $s_1, \ldots, s_n$  not all zero such that  $\overrightarrow{s_1 v_1} + \overrightarrow{s_2 v_2}$ Note: some of the  $s_i$  can be zero, just not all. 0

A set is linearly dependent if it is not linearly independent.

### A set of vectors is called linearly dependent if one vector is a linear

$$+ \cdots + s_n \overrightarrow{v_n} = 0$$

Equivalent definitions:

combination of the others.

• A set  $\{\vec{v_1}, \ldots, \vec{v_n}\}$  is **linearly independent** if the <u>only</u> solution to is  $s_1 = s_2 = \dots = s_n = 0$ .

A set is linearly independent if it is not linearly dependent.



### A set of vectors is called linearly independent if <u>no</u> vector is a linear

# $\overrightarrow{s_1v_1} + \overrightarrow{s_2v_2} + \cdots + \overrightarrow{s_nv_n} = \overrightarrow{0}$



0 Second Example:  $\left\{ \begin{bmatrix} 5\\24 \end{bmatrix}, \begin{bmatrix} 1\\0 \end{bmatrix}, \begin{bmatrix} 0\\1 \end{bmatrix} \right\}$  is linearly dependent.

Linearly independent: <u>no</u> vector is a linear combination of the others. 0 • Example:  $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$  is linearly independent.



Note that a single vector isn't called linearly dependent or independent. This is about sets of vectors.

Linearly dependent: one vector is a linear combination of the others.



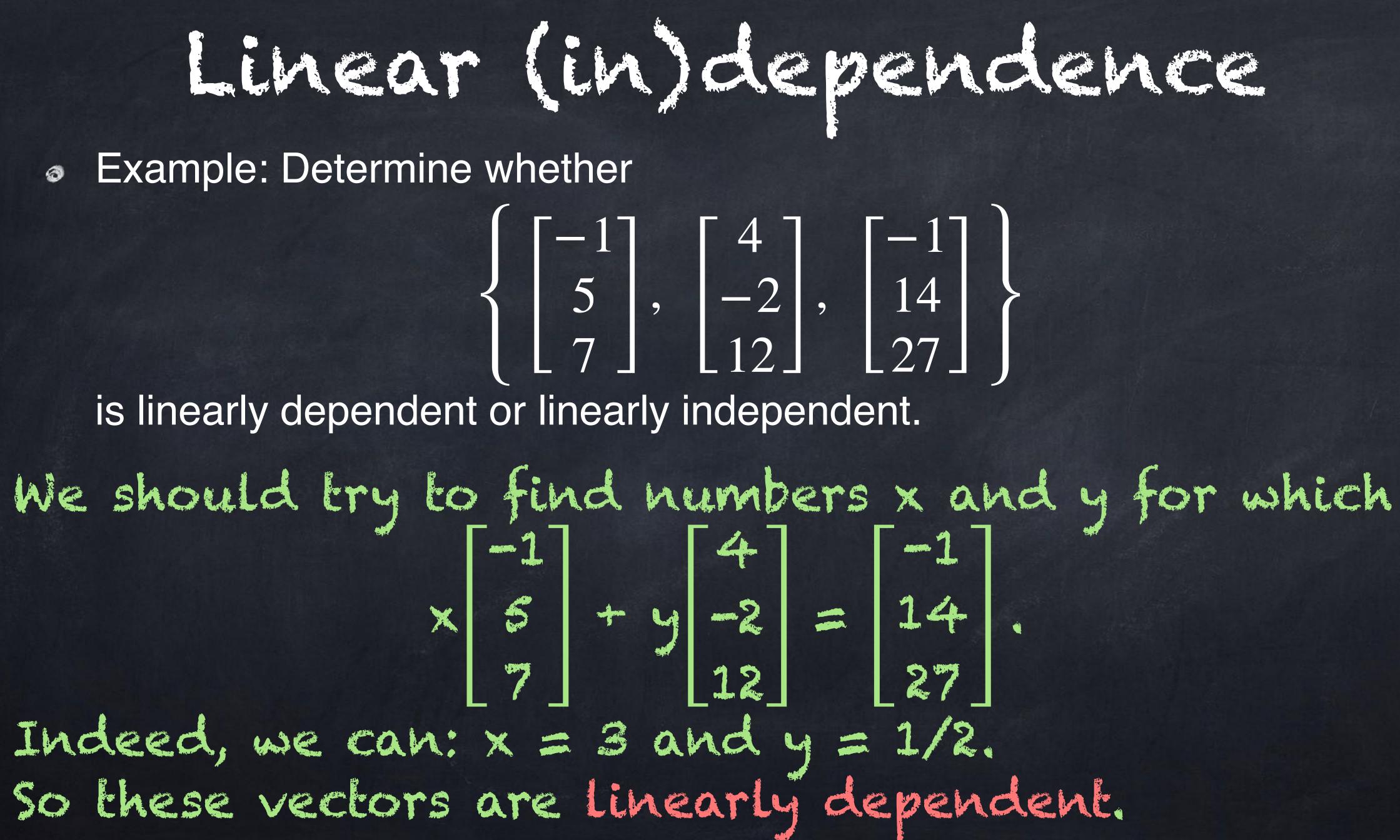
0 Sumple:  $\begin{bmatrix} 5 \\ 24 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$  are linearly dependent.

0 Second Example:  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$  are linearly independent.

Note that a single vector isn't called linearly dependent or independent. This is about sets of vectors.

Linearly dependent: one vector is a linear combination of the others.

Linearly independent: <u>no</u> vector is a linear combination of the others.

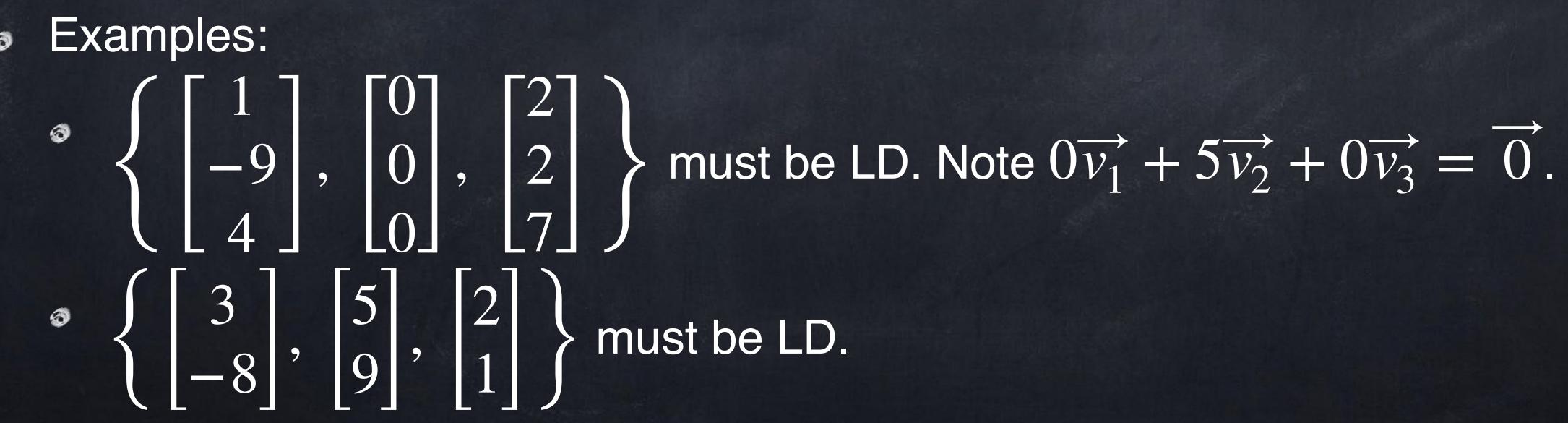




### Some facts to notice: 0

If a set contains the zero vector then it is linearly dependent.

of d+1 or more vectors will be linearly dependent.





- If the vectors are d-dimensional (each is a list of d numbers), then any set



### The rank of a matrix is the number of linearly independent columns.

An  $n \times m$  matrix can have rank at most  $\min(n, m)$ . An  $n \times m$  matrix is called full rank if its rank is equal to min(n, m).

