Math 1688

Thursday, 20 January 2022

Warm-up: Linear combinations.

theadamabrams.com/live

Last time: Linear combinations

A **linear combination** of some vectors is any sum of scalar multiples of those vectors.

- In symbols, $\overrightarrow{u}$ is a linear combination of $\overrightarrow{v}$ and $\overrightarrow{w}$ if
$$\overrightarrow{u} = s\overrightarrow{v} + t\overrightarrow{w}$$
for some numbers s, t.

- For more vectors, $\overrightarrow{u}$ is a linear combination of $\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots, \overrightarrow{v_n}$ if
$$\overrightarrow{u} = s_1\overrightarrow{v_1} + s_2\overrightarrow{v_2} + \cdots + s_n\overrightarrow{v_n}$$
for some numbers (scalars) $s_1, \ldots, s_n$.

- For one vector, $\overrightarrow{u}$ is a linear combination of $\overrightarrow{v}$ if $\overrightarrow{u} = s\overrightarrow{v}$.

Systems – summary so far

Any system of linear equations can be written as

$$A\overrightarrow{x} = \overrightarrow{b}.$$

coefficients → A; variables → $\overrightarrow{x}$; right-hand side → $\overrightarrow{b}$

- If A is square (same # of rows and cols) *and* $\det(A) \neq 0$, then the inverse matrix A^{-1} exists and the system has exactly one solution:

$$\overrightarrow{x} = A^{-1}\overrightarrow{b}.$$

- If A is square but $\det(A) = 0$, the system has either 0 or infinitely many solutions.
- If A is not square, there is no determinant or inverse.

rank(A) will help us determine the number of solutions in these cases.

Rank

The **rank** of a matrix is the maximum number of linearly independent rows.

- Remember that a set of vectors is **linearly independent** if no vector is a linear combination of the others.
  - Remember that a **linear combination** of vectors is any sum of scalar multiples of the vectors: $a\vec{v} + b\vec{w} + \cdots$

max. # of lin. indep. rows = max. # of lin. indep. columns

An $n \times m$ matrix can have rank at most $\min(n, m)$.
An $n \times m$ matrix is called **full rank** if its rank is equal to $\min(n, m)$.

Rank

The **rank** of a matrix is the maximum number of linearly independent rows (and also the maximum number of linearly independent columns—these will always be the same number!).

Example: What is the rank of $\begin{bmatrix} 3 & 1 & -9 & 0 & 6 \\ 2 & 0 & 4 & 1 & -3 \end{bmatrix}$? rank 2

Example: What is the rank of $\begin{bmatrix} -9 & 18 \\ 2 & -4 \\ 5 & -10 \end{bmatrix}$? rank 1

because $\begin{bmatrix} 18 \\ -4 \\ -10 \end{bmatrix} = -2 \begin{bmatrix} -9 \\ 2 \\ 5 \end{bmatrix}$

Rank

The **rank** of a matrix is the maximum number of linearly independent rows (and also the maximum number of linearly independent columns—these will always be the same number!).

Example: What is the rank of $\begin{bmatrix} 5 & 2 & 7 \\ 1 & 0 & 1 \\ 12 & 7 & 19 \end{bmatrix}$?

rank 2

because $\begin{bmatrix} 7 \\ 1 \\ 19 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 12 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 7 \end{bmatrix}$

Example: What is the rank of $\begin{bmatrix} 5 & 2 & 7 \\ 1 & 0 & 1 \\ 12 & 7 & 18 \end{bmatrix}$?

rank 3

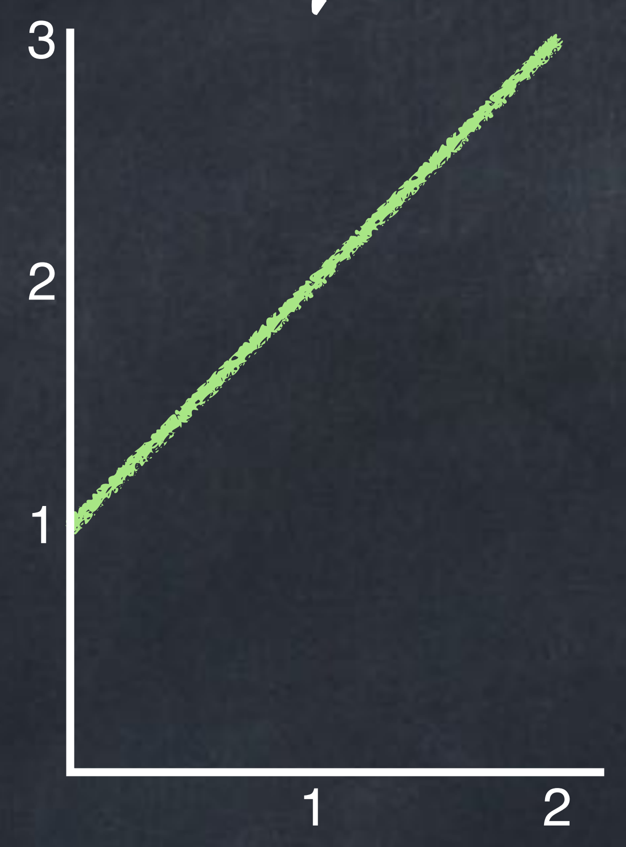
because $\begin{bmatrix} 7 \\ 1 \\ 19 \end{bmatrix} = a\begin{bmatrix} 5 \\ 1 \\ 12 \end{bmatrix} + b\begin{bmatrix} 2 \\ 0 \\ 7 \end{bmatrix}$ is impossible

Rank as amount of information

$$\begin{cases} -x + y = 1 \end{cases}$$

$$\begin{bmatrix} -1 & 1 & 1 \end{bmatrix}$$

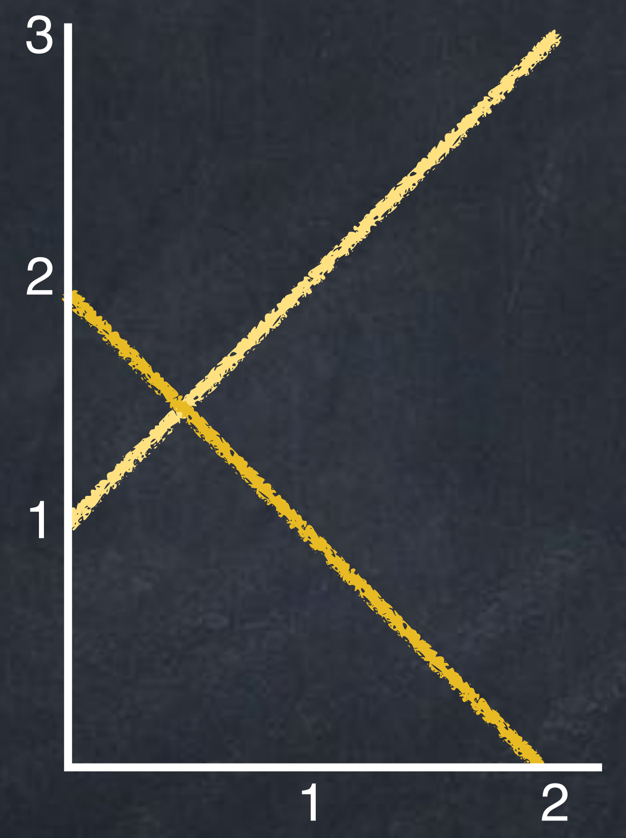
has rank 1



$$\begin{cases} -x + y = 1 \\ x + y = 2 \end{cases}$$

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

has rank 2

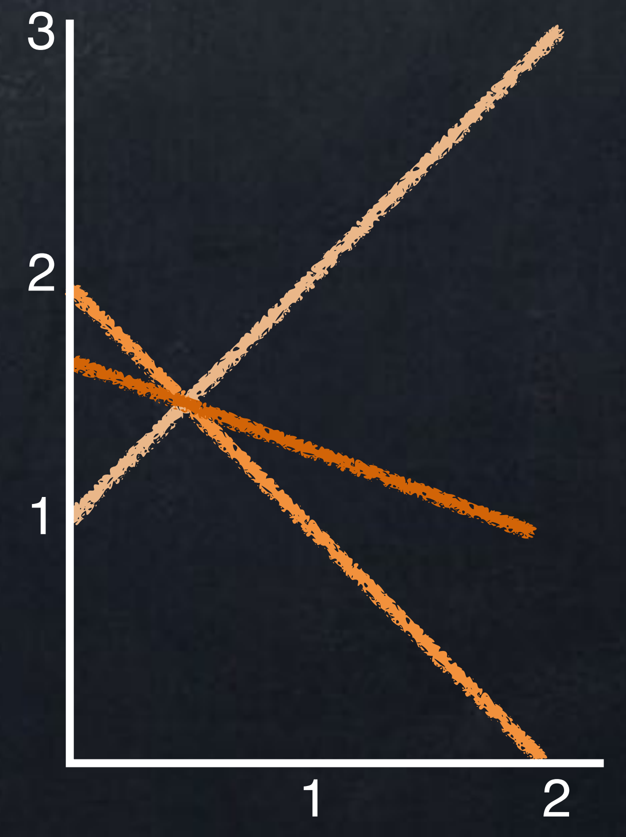


$$\begin{cases} -x + y = 1 \\ x + y = 2 \\ x + 3y = 5 \end{cases}$$

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 3 & 5 \end{bmatrix}$$

has rank 2 also

No new info!



Rank as amount of information

$$\begin{cases} x + y + z = 6 \\ x + y \qquad\; = 3 \end{cases}$$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 1 & 0 & 3 \end{bmatrix}$$

has rank 2

Can we say anything about $x + y - 3z$?

$\times(-3)$

$\times 4$

$$-3x - 3y - 3z = -18$$

$$+ \,(4x + 4y \qquad = 12)$$

$$\overline{x + y - 3z = -6}$$

Rank as amount of information

$$\begin{cases} x + y + z = 6 \\ x + y \qquad = 3 \end{cases}$$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 1 & 0 & 3 \end{bmatrix}$$

has rank 2

$$\begin{cases} x + y + \ \ z = 6 \\ x + y \qquad = 3 \\ x + y - 3z = -6 \end{cases}$$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 1 & 0 & 3 \\ 1 & 1 & -3 & -6 \end{bmatrix}$$

also has rank 2

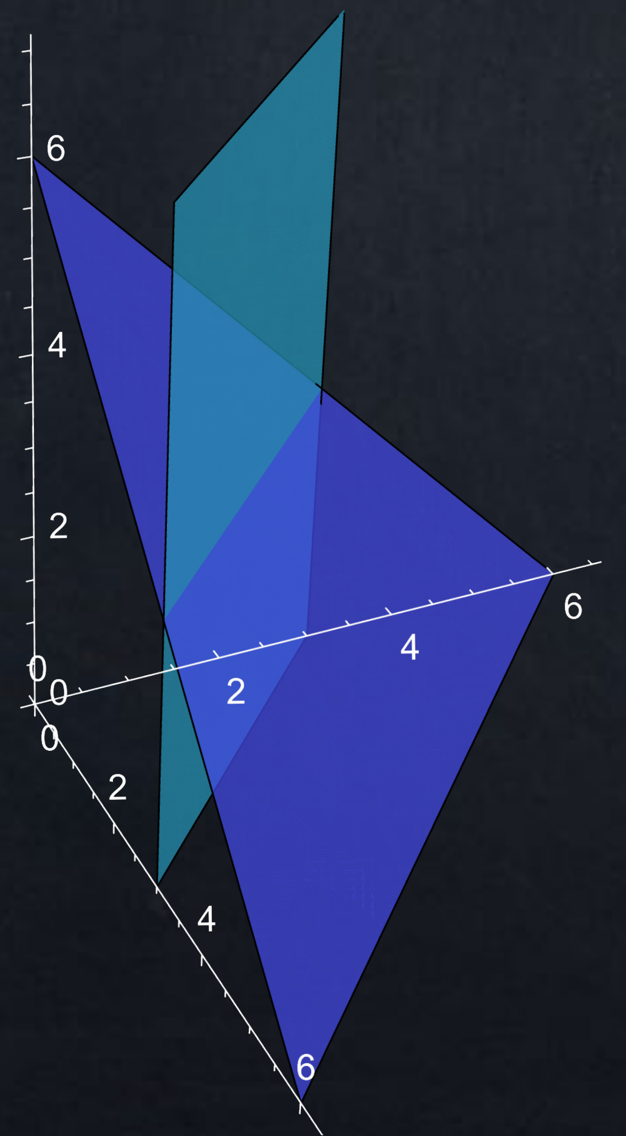
No new information!

Rank as amount of information

$$\begin{cases} x+y+z=6 \\ x+y \quad\quad = 3 \end{cases}$$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 1 & 0 & 3 \end{bmatrix}$$

has rank 2



$$\begin{cases} x+y+z=6 \\ x+y \quad\quad = 3 \\ x+y-3z=-6 \end{cases}$$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 1 & 0 & 3 \\ 1 & 1 & -3 & -6 \end{bmatrix}$$

has rank 2

No new info!

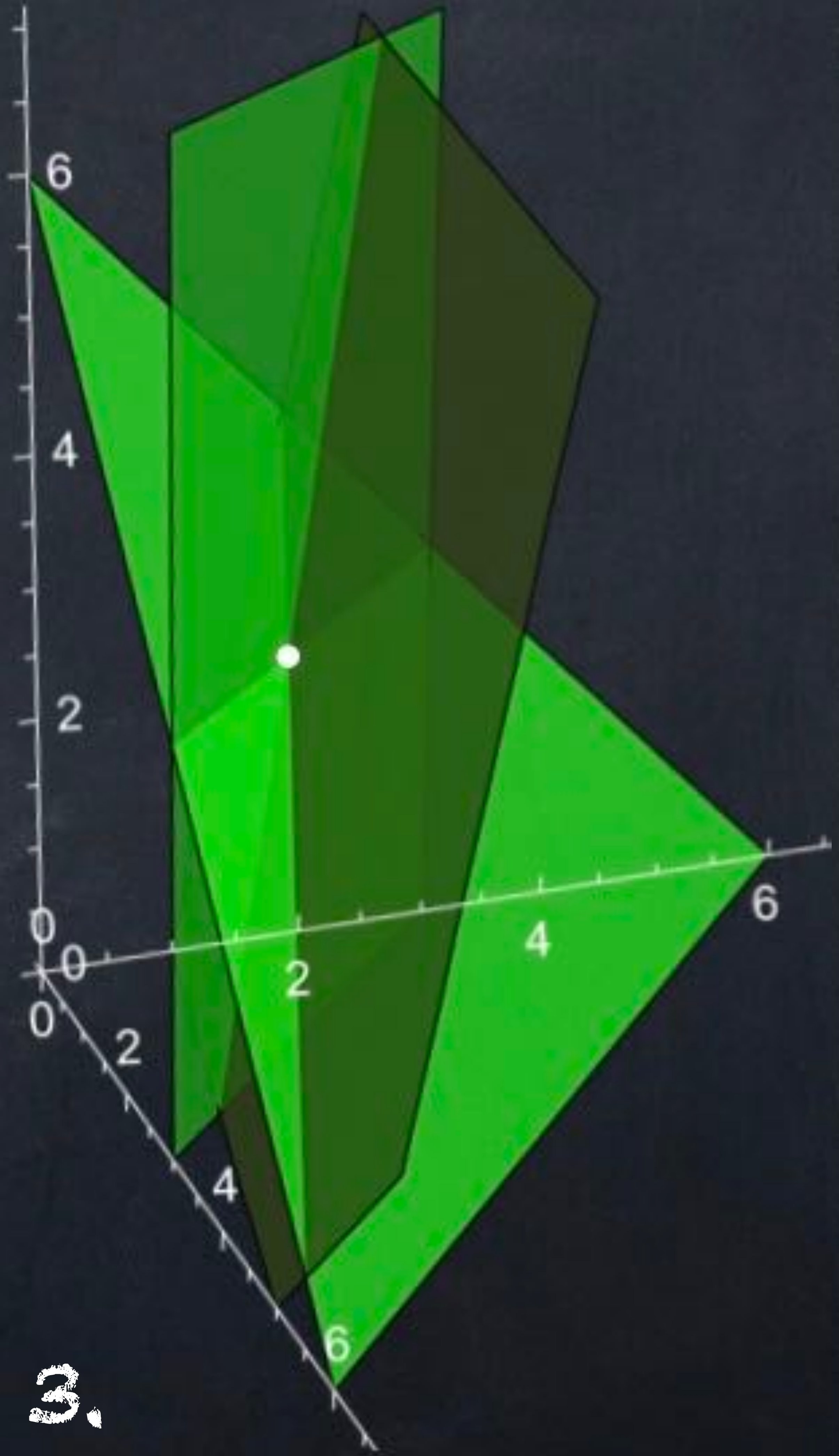
$$\begin{cases} x+y+z=6 \\ x+y \quad\quad = 3 \\ x+5y-z=5 \end{cases}$$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 1 & 0 & 3 \\ 1 & 5 & -1 & 5 \end{bmatrix}$$

has rank 3

New information!

Matrix now has rank 3.



Rank as amount of information

$$\begin{cases} x + y + z = 6 \\ x + y \phantom{{}+z} = 3 \end{cases}$$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 1 & 0 & 3 \end{bmatrix}$$

has rank 2

$$\begin{cases} x + y + z = 6 \\ x + y \phantom{{}-3z} = 3 \\ x + y - 3z = -6 \end{cases}$$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 1 & 0 & 3 \\ 1 & 1 & -3 & -6 \end{bmatrix}$$

has rank 2

$$\begin{cases} x + y + z = 6 \\ x + y \phantom{{}-z} = 3 \\ x + 5y - z = 5 \end{cases}$$

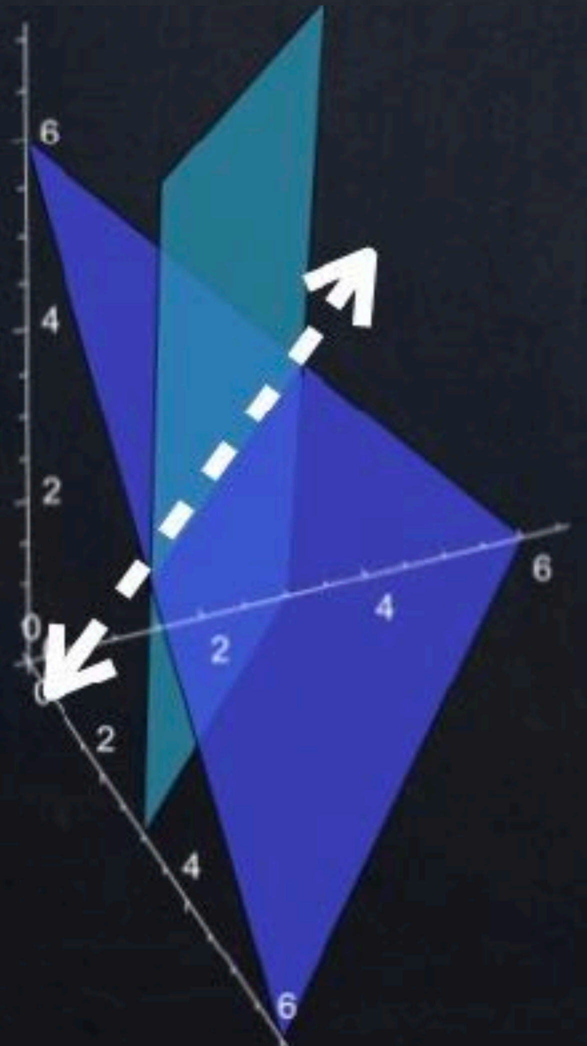
$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 1 & 0 & 3 \\ 1 & 5 & -1 & 5 \end{bmatrix}$$

has rank 3

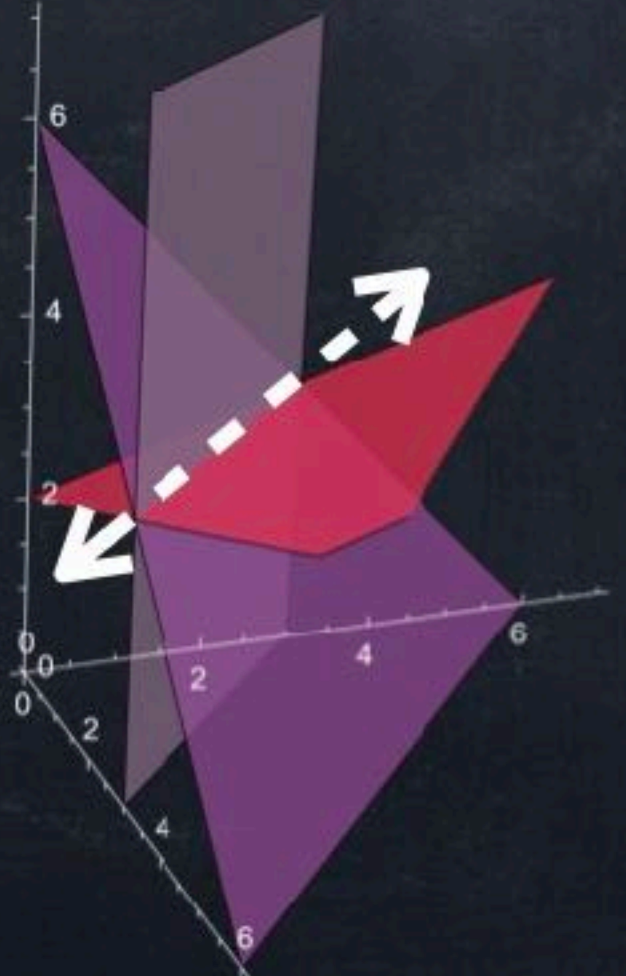
$$\begin{cases} x + y + z = 6 \\ x + y = 3 \\ 5x + 5y = 10 \end{cases}$$

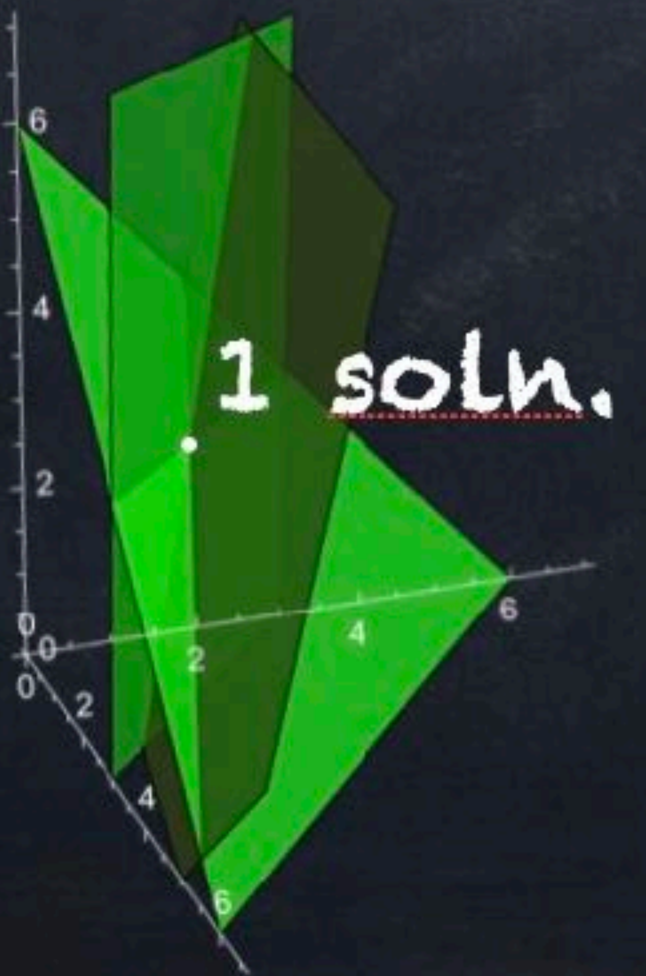
$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 1 & 0 & 3 \\ 5 & 5 & 0 & 10 \end{bmatrix}$$

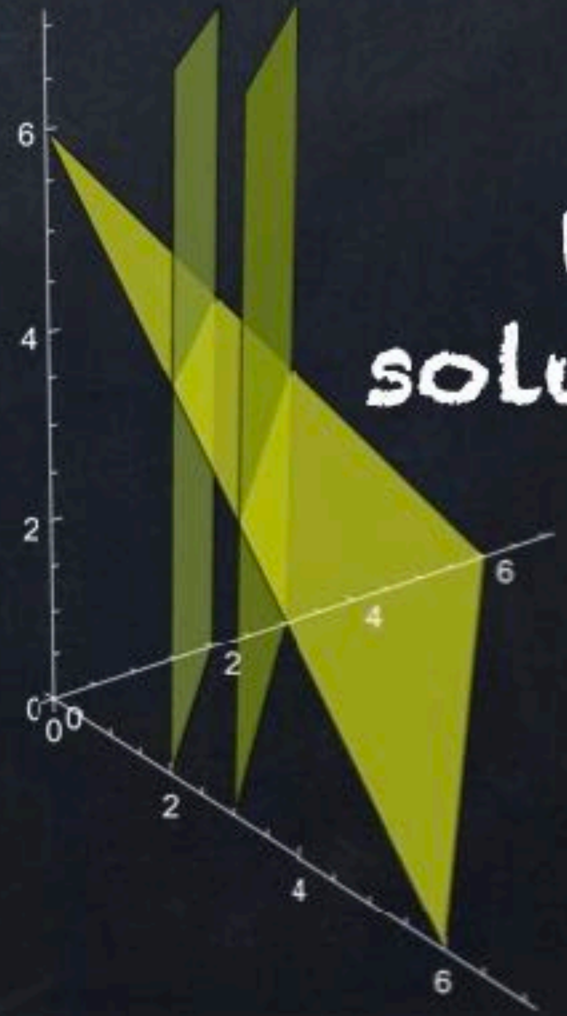
has rank 3



infinitely many solutions







no solutions

Augmented matrix

For a system $A\vec{x} = \vec{b}$, the matrix A is called the "coefficient matrix".

The **augmented matrix** for the system is the matrix formed by adding column $\vec{b}$ to the matrix A. We write $[A \mid \vec{b}]$ for this matrix.

- Example: For the system

$$\begin{cases} 4x + 9y = 6 \\ 2y + 3z = 0 \end{cases}$$

we have

$$A = \begin{bmatrix} 4 & 9 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad [A \mid \vec{b}] = \begin{bmatrix} 4 & 9 & 6 \\ 2 & 3 & 0 \end{bmatrix}.$$

Often written

$$\left[\begin{array}{cc|c} 4 & 9 & 6 \\ 2 & 3 & 0 \end{array}\right]$$

with a | before the last column.

Augmented matrix

For a system $A\overrightarrow{x} = \overrightarrow{b}$, the matrix A is called the "coefficient matrix".

The **augmented matrix** for the system is the matrix formed by adding column $\overrightarrow{b}$ to the matrix A. We write $\left[A \mid \overrightarrow{b}\right]$ for this matrix.

The Rouché–Capelli Theorem

The system $A\overrightarrow{x} = \overrightarrow{b}$ is consistent if and only if $\operatorname{rank}(A) = \operatorname{rank}\left([A \mid \overrightarrow{b}]\right)$. If it is consistent, the collection of all solutions has dimension $n - \operatorname{rank}(A)$, where n is the number of variables.

Augmented matrix

The Rouché–Capelli Theorem

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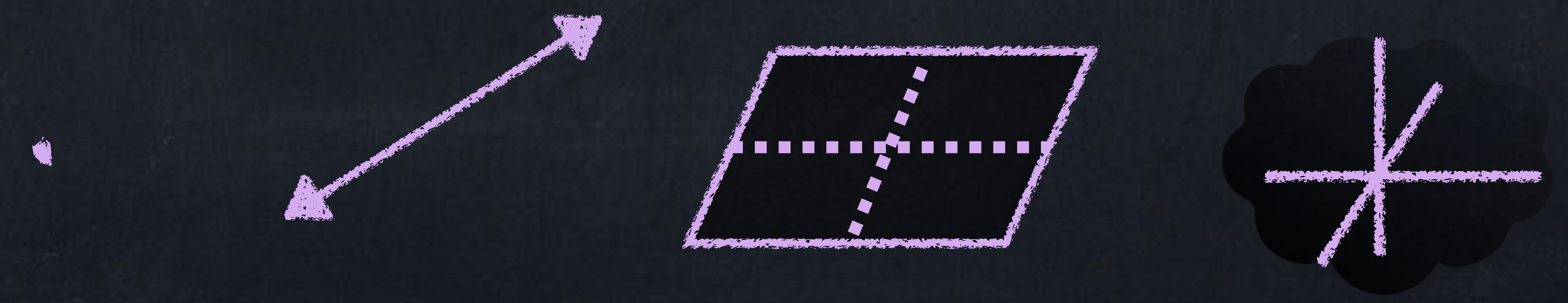
Dimension: 0 1 2 3



Rank/system examples

Ex 1.

$$\begin{cases} 5x + 2y + 7z = 6 \\ x + z = 4 \\ 12x + 7y + 18z = 9 \end{cases}$$

$$A = \begin{bmatrix} 5 & 2 & 7 \\ 1 & 0 & 1 \\ 12 & 7 & 18 \end{bmatrix}$$

$$[A \mid \vec{b}\,] = \left[\begin{array}{ccc|c} 5 & 2 & 7 & 6 \\ 1 & 0 & 1 & 4 \\ 12 & 7 & 18 & 9 \end{array}\right]$$

rank(A) = 3

rank(A|b) = 3

The coeff. and augmented matrices have the same rank, so the system does have at least one solution. The space of all solutions has dimension

(# of variables) - (rank of A) = 3 - 3 = 0,

so the set of solutions is just one point.

Rank/system examples

Ex 2.

$$\begin{cases} 5x + 2y + 7z = 6 \\ x + z = 4 \\ 12x + 7y + 19z = 9 \end{cases}$$

$$A = \begin{bmatrix} 5 & 2 & 7 \\ 1 & 0 & 1 \\ 12 & 7 & 19 \end{bmatrix}$$

$$[A \mid \vec{b}] = \left[\begin{array}{ccc|c} 5 & 2 & 7 & 6 \\ 1 & 0 & 1 & 4 \\ 12 & 7 & 19 & 9 \end{array}\right]$$

rank(A) = 2

rank(A|b) = 3

The coefficient and augmented matrices have different ranks, so there are no solutions to the system.

Rank/system examples

Ex 3.

$$\begin{cases} 5x + 2y + 7z = \underline{10} \\ x + z = \underline{4} \\ 12x + 7y + 19z = \underline{13} \end{cases}$$

$$A = \begin{bmatrix} 5 & 2 & 7 \\ 1 & 0 & 1 \\ 12 & 7 & 19 \end{bmatrix}$$

$$[A \mid \vec{b}] = \left[\begin{array}{ccc|c} 5 & 2 & 7 & \underline{10} \\ 1 & 0 & 1 & \underline{4} \\ 12 & 7 & 19 & \underline{13} \end{array}\right]$$

rank(A) = 2

rank(A|b) = 2

The coeff. and augmented matrices have the same rank, so the system does have at least one solution. The space of all solutions has dimension

(# of variables) - (rank of A) = 3 - 2 = 1,

so the set of solutions is a LINE in 3D space.

Free variables

How can we describe the solutions nicely when there are infinitely many?

A **free variable** is a variable whose value can be set to anything when describing solutions to a system.

- If $n - \text{rank}(A) = d$, we have d free variables.
- We can choose which of the variables are free.

Free variables

Ex 3 again

$$\begin{cases} 5x + 2y + 7z = 10 \\ x + z = 4 \\ 12x + 7y + 19z = 13 \end{cases}$$

rank(A) = 2

rank(A|b) = 2

(# of vars.)-rank(A) = 1

We know we have exactly one free variable.
We can pick any one of x or y or z for that variable.

With x free, all solutions look like (x, x-9, 4-x)

With y free: (x,y,z) = (y+9, y, -y-5)

With z free: (x,y,z) = (4-z, -5-z, z)

Rank and determinant

$$\begin{cases} 5x + 2y + 7z = 6 \\ x + z = 4 \\ 12x + 7y + 18z = 9 \end{cases} \qquad A = \begin{bmatrix} 5 & 2 & 7 \\ 1 & 0 & 1 \\ 12 & 7 & 18 \end{bmatrix}$$

$\det(A) \neq 0$

$\operatorname{rank}(A) = 3$

$n - \operatorname{rank}(A) = 0$

$$\begin{cases} 5x + 2y + 7z = 6 \\ x + z = 4 \\ 12x + 7y + 19z = 9 \end{cases}$$

$$\begin{cases} 5x + 2y + 7z = 10 \\ x + z = 4 \\ 12x + 7y + 19z = 13 \end{cases}$$

$$A = \begin{bmatrix} 5 & 2 & 7 \\ 1 & 0 & 1 \\ 12 & 7 & 19 \end{bmatrix}$$

$\det(A) = 0$

$\operatorname{rank}(A) = 2$

For an $n \times n$ matrix A, $\det(A) = 0$ if and only if $\operatorname{rank}(A) < n$.

Right-hand side zeros

Using what we know about determinant, rank, etc., what can we say about a square system where the right-side has only zeros?

$$\begin{cases} 5x + z = 0 \\ 2x + 2y + 3z = 0 \\ -8x + 2y + z = 0 \end{cases}$$

- $(x, y, z) = (0,0,0)$ is definitely a solution.
- In order to have any *other* solution, the coefficient matrix must have a determinant of 0.
  - In that case there will be infinitely many solutions (the set of all solutions will form a line or a plane in 3D space).

Right-hand side zeros

Using what we know about determinant, rank, etc., what can we say about a square system where the right-side has only zeros?

- There is at least one solution: all variables could be 0.
  As a vector, this is $\overrightarrow{x} = [x, y, ...] = \overrightarrow{0}$.
- Can there be other solutions?

If $M\overrightarrow{x} = \overrightarrow{0}$ has solutions other than $\overrightarrow{x} = \overrightarrow{0}$, then $\det(M) = 0$.

Transformations

There are many other applications of matrices besides systems of equations. One of the most common is visual “transformations”:

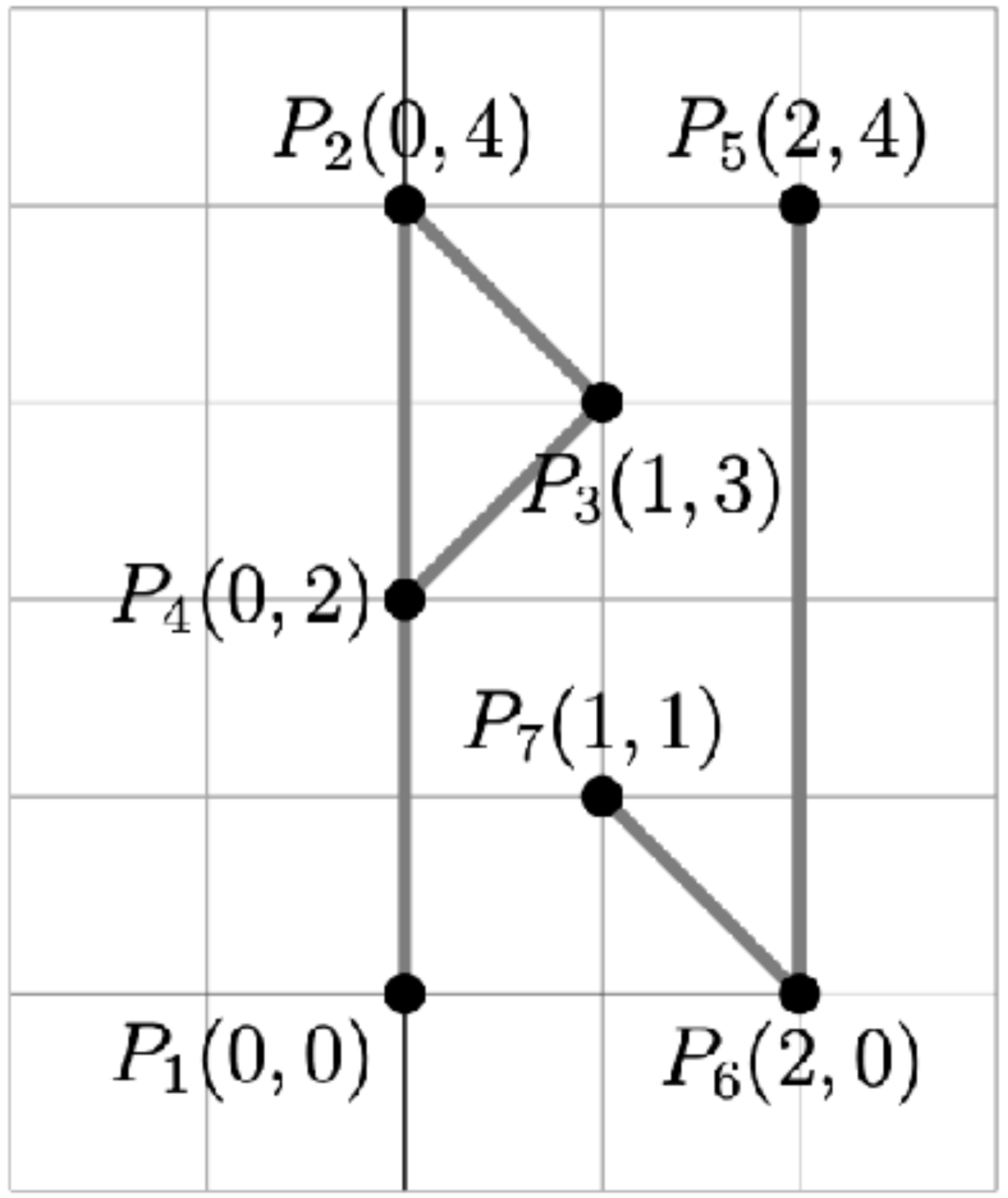
$$\begin{bmatrix} x_{\text{new}} \\ y_{\text{new}} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_{\text{old}} \\ y_{\text{old}} \end{bmatrix}.$$

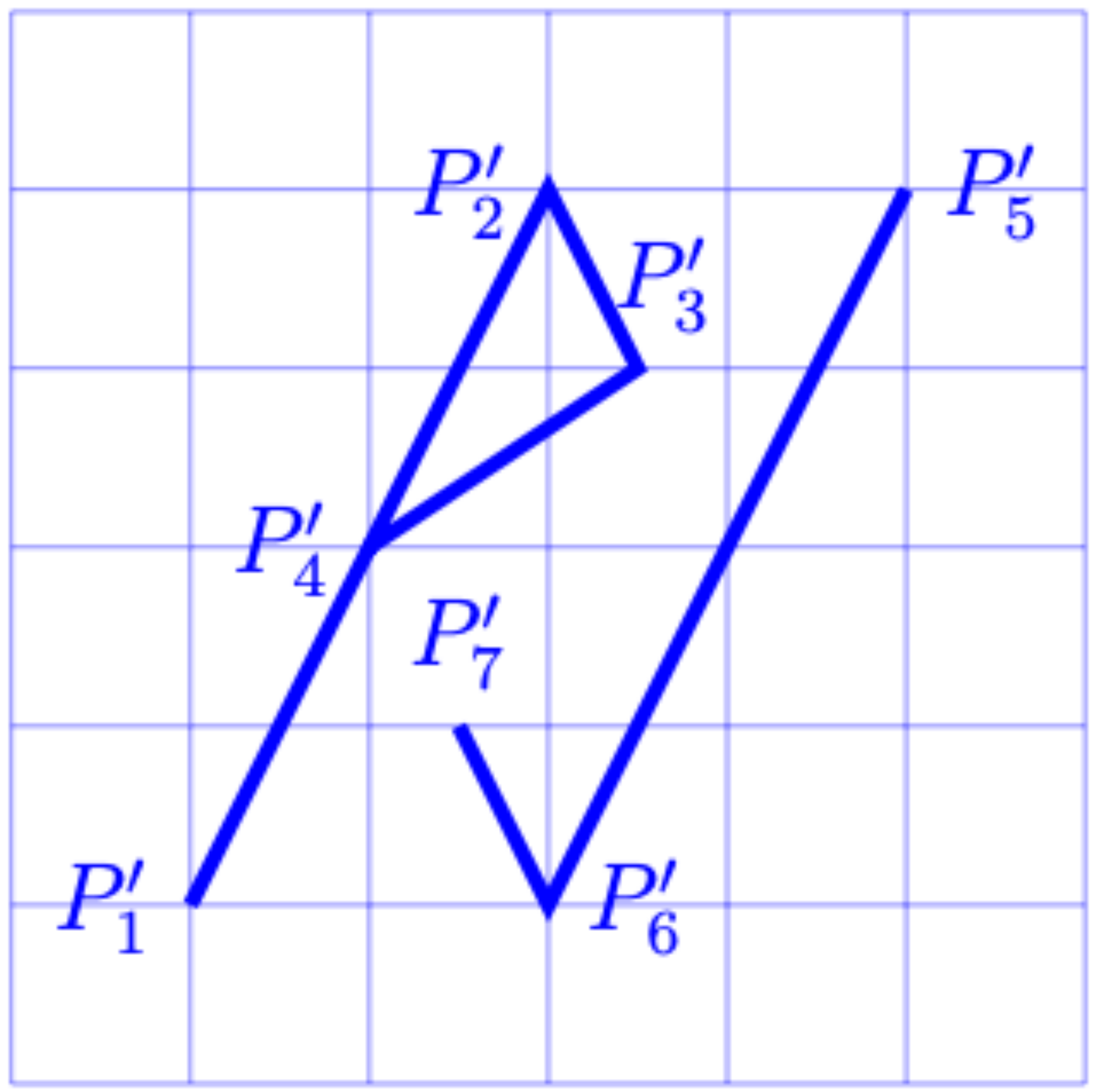
86. For each of the points P_1 through P_7, calculate

$$P_i' = \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix} P_i.$$

(For example, for $P_5' = \left[\begin{smallmatrix} 1 & 1/2 \\ 0 & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 2 \\ 4 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 4 \\ 4 \end{smallmatrix}\right]$.) Plot the points $P_1', \ldots, P_7'$ on a new grid. Connect $P_1' \to P_2' \to P_3' \to P_4'$ with line segments, and connect $P_5' \to P_6' \to P_7'$.

Congratulations. You can write italic.





Transformations

There are many other applications of matrices besides systems of equations. One of the most common is visual "transformations":

$$\begin{bmatrix} x_{\text{new}} \\ y_{\text{new}} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_{\text{old}} \\ y_{\text{old}} \end{bmatrix}.$$

Multiplying by a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ moves points around.

- Could there be a point that doesn't move?
- Could there be a collection of points that, after moveming each point in it, is still the same collection?
  - Answers: Yes, but if one point (other than the origin) is fixed then so is an entire line containing it.

Eigenvectors and eigenvalues

For a square matrix A, if we have

$$A\vec{v} = \lambda\vec{v}$$

for some number λ and some vector $\vec{v} \neq \vec{0}$ then

- the vector $\vec{v}$ is called an **eigenvector** of A, and
- the number λ is called an **eigenvalue** of A.

Note that if $\vec{v}$ is an eigenvector, any scalar multiple of $\vec{v}$ will also be an eigenvector.

Finding eigenvalues

How could we find the eigenvalues of $\begin{bmatrix} 6 & 4 \\ 1 & 3 \end{bmatrix}$?

We want $\begin{bmatrix} 6 & 4 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \lambda\begin{bmatrix} x \\ y \end{bmatrix}$ for some x, y, λ.

So $\begin{cases} 6x+4y = \lambda x \\ x+3y = \lambda y \end{cases}$ and so $\begin{cases} (6-\lambda)x + 4y = 0 \\ x+(3-\lambda)y = 0 \end{cases}$

If $M\vec{x} = \vec{0}$ has solutions other than $\vec{x} = \vec{0}$, then $\det(M) = 0$.

For this system to have solutions other than $(x,y) = (0,0)$, we must have $\det\left(\begin{bmatrix} 6-\lambda & 4 \\ 1 & 3-\lambda \end{bmatrix}\right) = (6-\lambda)(3-\lambda) - 4(1) = 0.$

polynomial $\lambda^2 - 9\lambda + 14 = 0$

gives us $\lambda = 2$ or $\lambda = 7$

Finding eigenvalues

The system

$$\begin{cases} (a-\lambda)x + \quad by = 0 \\ cx + (d-\lambda)y = 0 \end{cases}$$

has a non-zero solution exactly when $\det\left(\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}\right) = 0.$

In general (including for larger square matrices),
the eigenvalues of A are the values of λ for which $\det(A - \lambda I) = 0.$
Here I is the identity matrix of the same dimensions as A.

Finding eigenvalues

The eigenvalues of A are the values of λ for which $\det(A - \lambda I) = 0$.

Algebra proof: if $A\overrightarrow{v} = \lambda\overrightarrow{v}$ for some $\overrightarrow{v} \neq \overrightarrow{0}$ then

$$A\overrightarrow{v} = I\left(\lambda\overrightarrow{v}\right)$$

$$A\overrightarrow{v} - (\lambda I)\overrightarrow{v} = \overrightarrow{0}$$

$$(A - \lambda I)\overrightarrow{v} = \overrightarrow{0}$$

$$\det(A - \lambda I) = 0$$

Finding eigenvectors

Knowing 7 is an eigenvalue, how do we find the eigenvectors of $\begin{bmatrix} 6 & 4 \\ 1 & 3 \end{bmatrix}$ associated to 7?

We want $\begin{bmatrix} 6 & 4 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 7\begin{bmatrix} x \\ y \end{bmatrix}$ for some x,y.

So $\begin{cases} 6x+4y = 7x \\ x+3y = 7y \end{cases}$ and so $\begin{cases} -x + 4y = 0 \\ x-4y = 0 \end{cases}$

This system has one free variable. All solutions are of the form $(x, x/4)$, so any scalar multiple of $[1, 1/4]$ is an eigenvector.

Finding eigenvectors

Knowing that 7 is an eigenvalue of $\begin{bmatrix} 6 & 4 \\ 1 & 3 \end{bmatrix}$, we get that if $\overrightarrow{v}$ is any scalar multiple of $\begin{bmatrix} 1 \\ 1/4 \end{bmatrix}$ then $\begin{bmatrix} 6 & 4 \\ 1 & 3 \end{bmatrix}\overrightarrow{v} = 7\overrightarrow{v}$.

Similarly, eigenvalue 2 leads to scalar multiples of $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

- We can say, "$\begin{bmatrix} 1 \\ 1/4 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are the eigenvectors of $\begin{bmatrix} 6 & 4 \\ 1 & 3 \end{bmatrix}$".
- But we can really use any scalar multiples. So we could also say "$\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -29 \\ 29 \end{bmatrix}$ are the eigenvectors of $\begin{bmatrix} 6 & 4 \\ 1 & 3 \end{bmatrix}$" and be equally correct.