Math 1688W

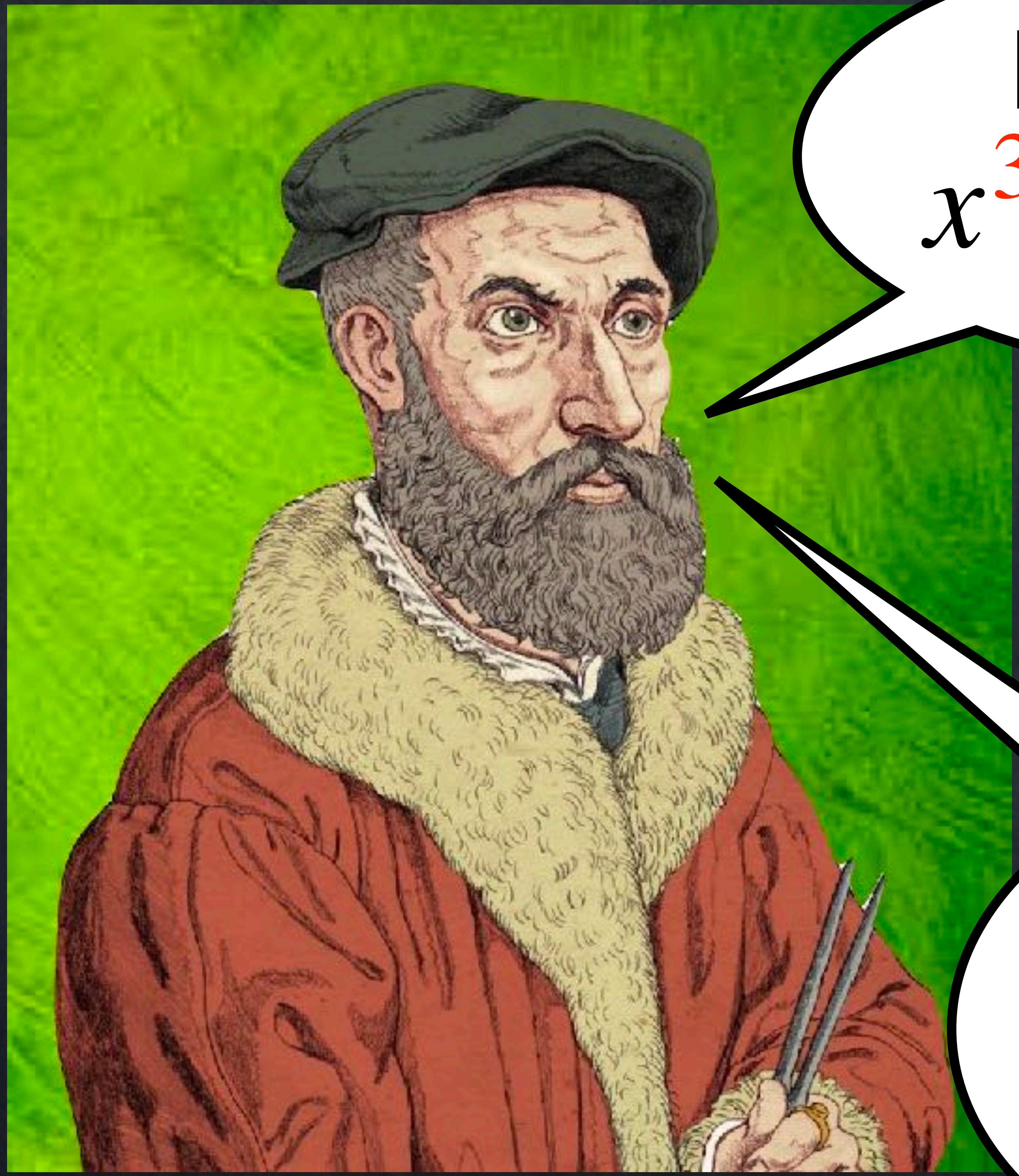
Thursday, 14 October

Do this now.

Warm-up:
Calculate i^2, i^3, i^4, i^5, i^6, and i^7.

Warm-up 2:
$(r \cdot 2^k)^n = \underline{r^n} \cdot \underline{2^{kn}}$

History of complex #s

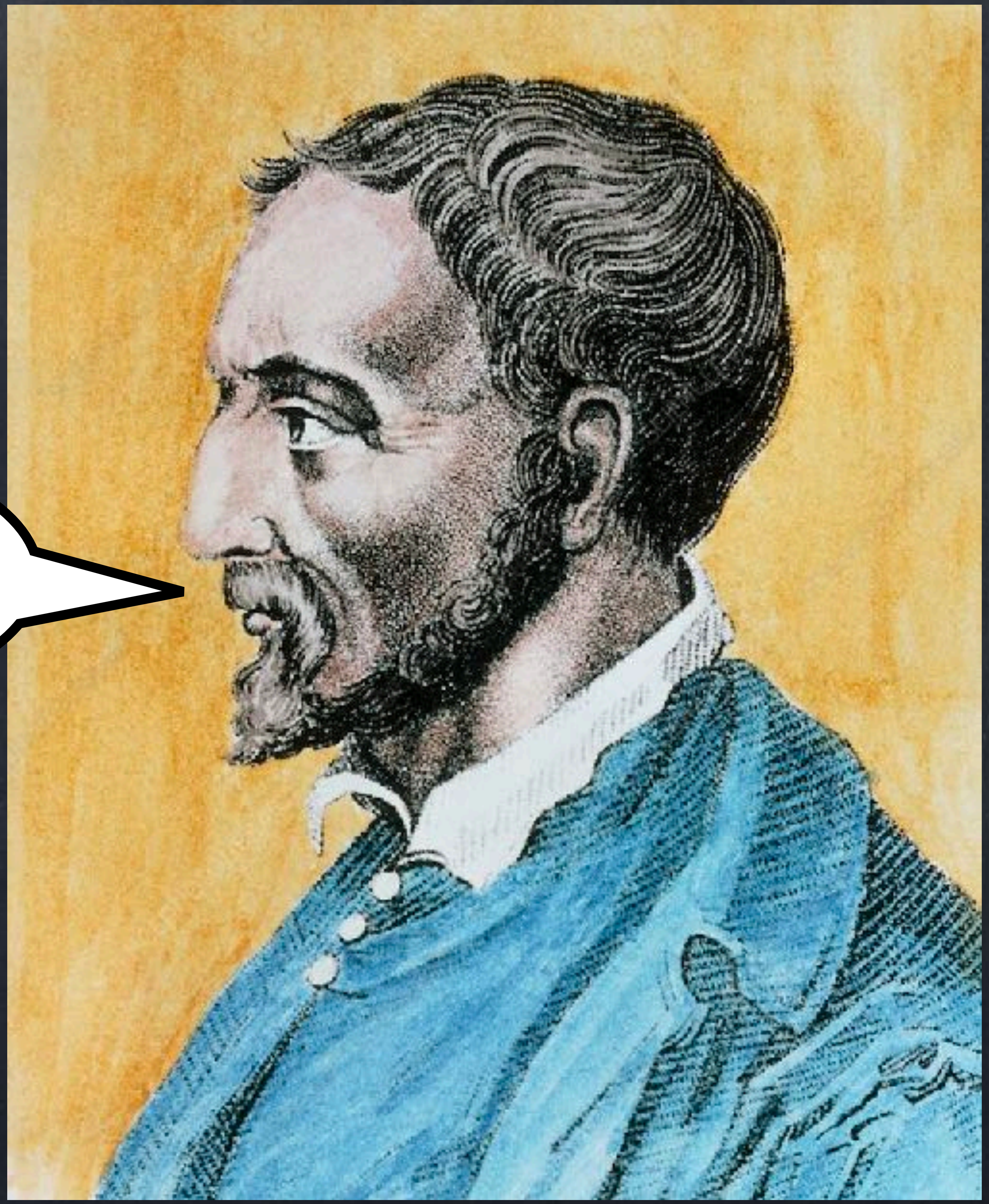


Niccolò Tartaglia
1500 - 1557

I can solve $x^3 + mx = n$.

How?!

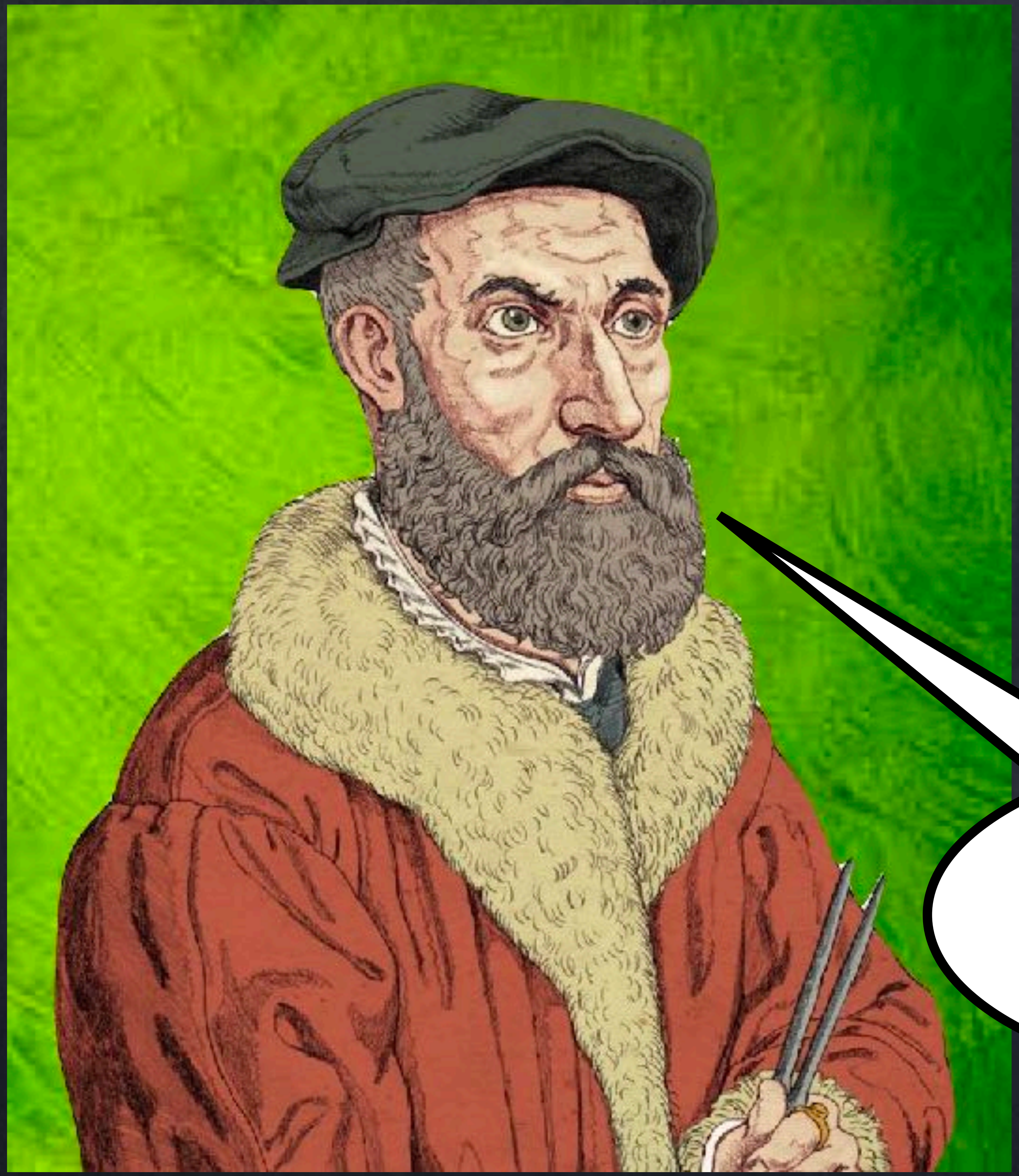
I will tell you, but you can't tell anyone else.



Gerolamo Cardano
1502 - 1576

Art: Matteo Bergamelli, Science Photo Library

History of complex #s



Niccolò Tartaglia
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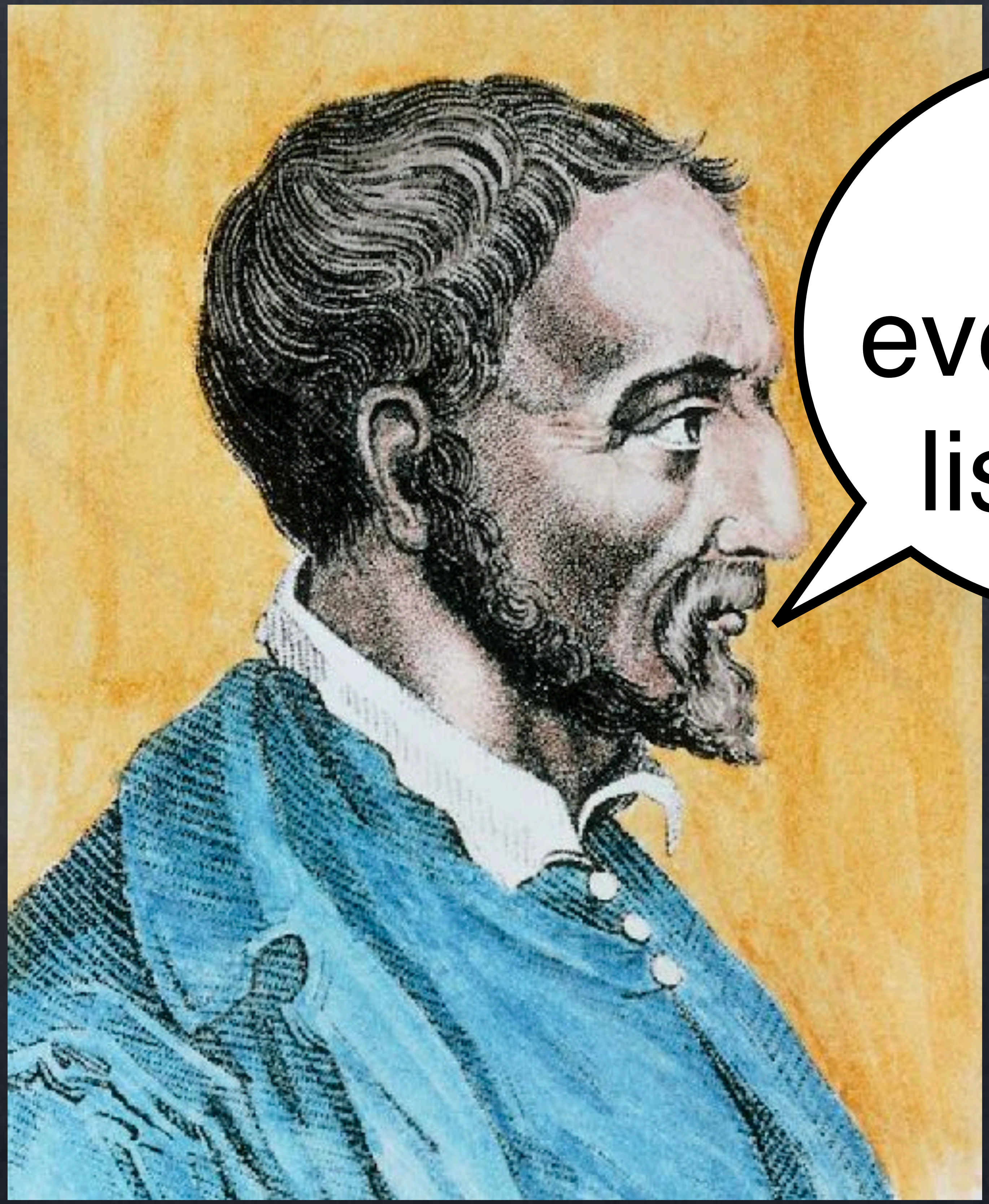


Ars Magna
(1545)

Gerolamo Cardano
1502 - 1576

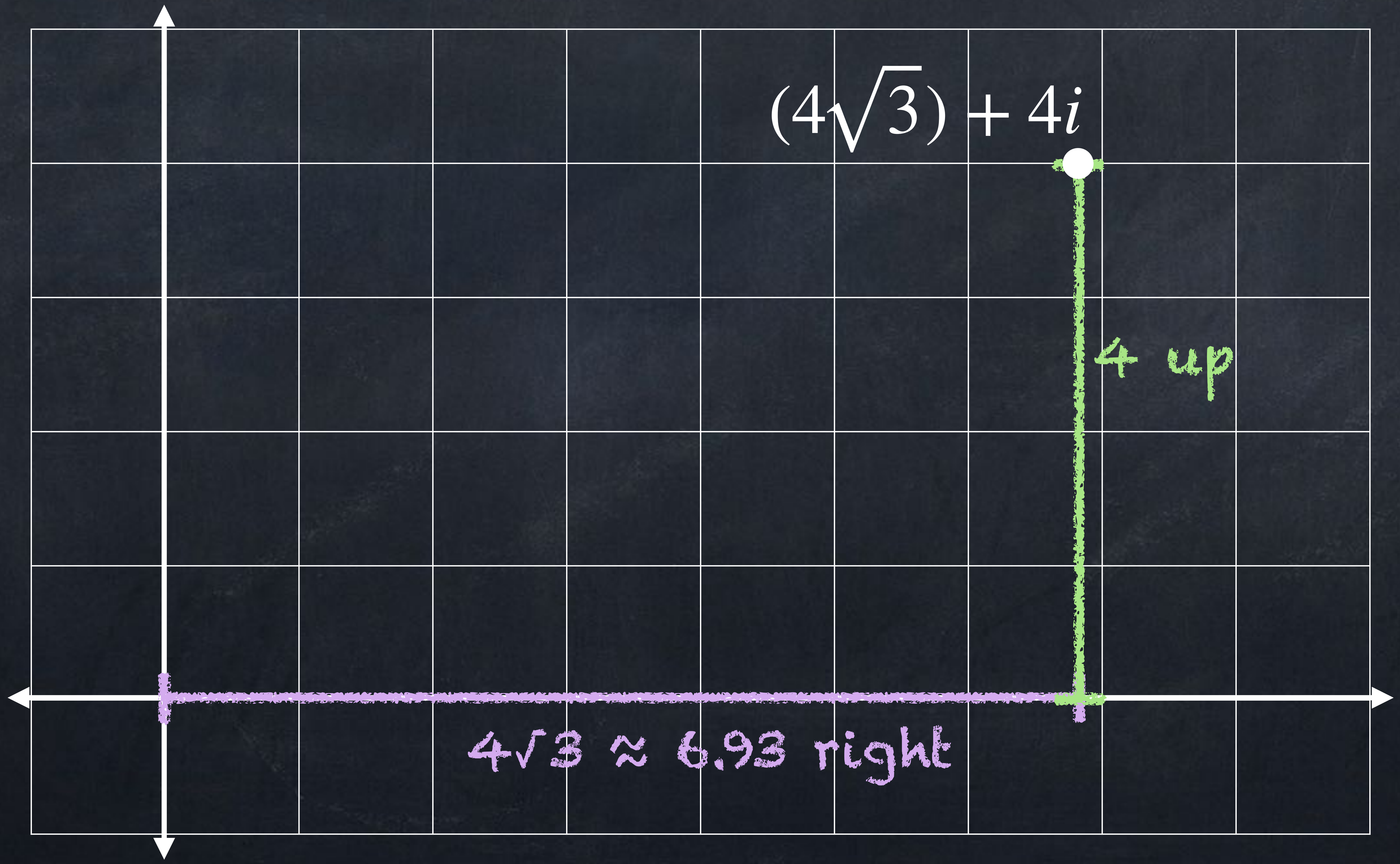
Art: Matteo Bergamelli, Science Photo Library

Where is the point

$$z = 8\cos(\pi/6) + 8\sin(\pi/6)\,i$$

on the complex plane?

$$z = 8(\sqrt{3}/2) + 8(1/2)i$$



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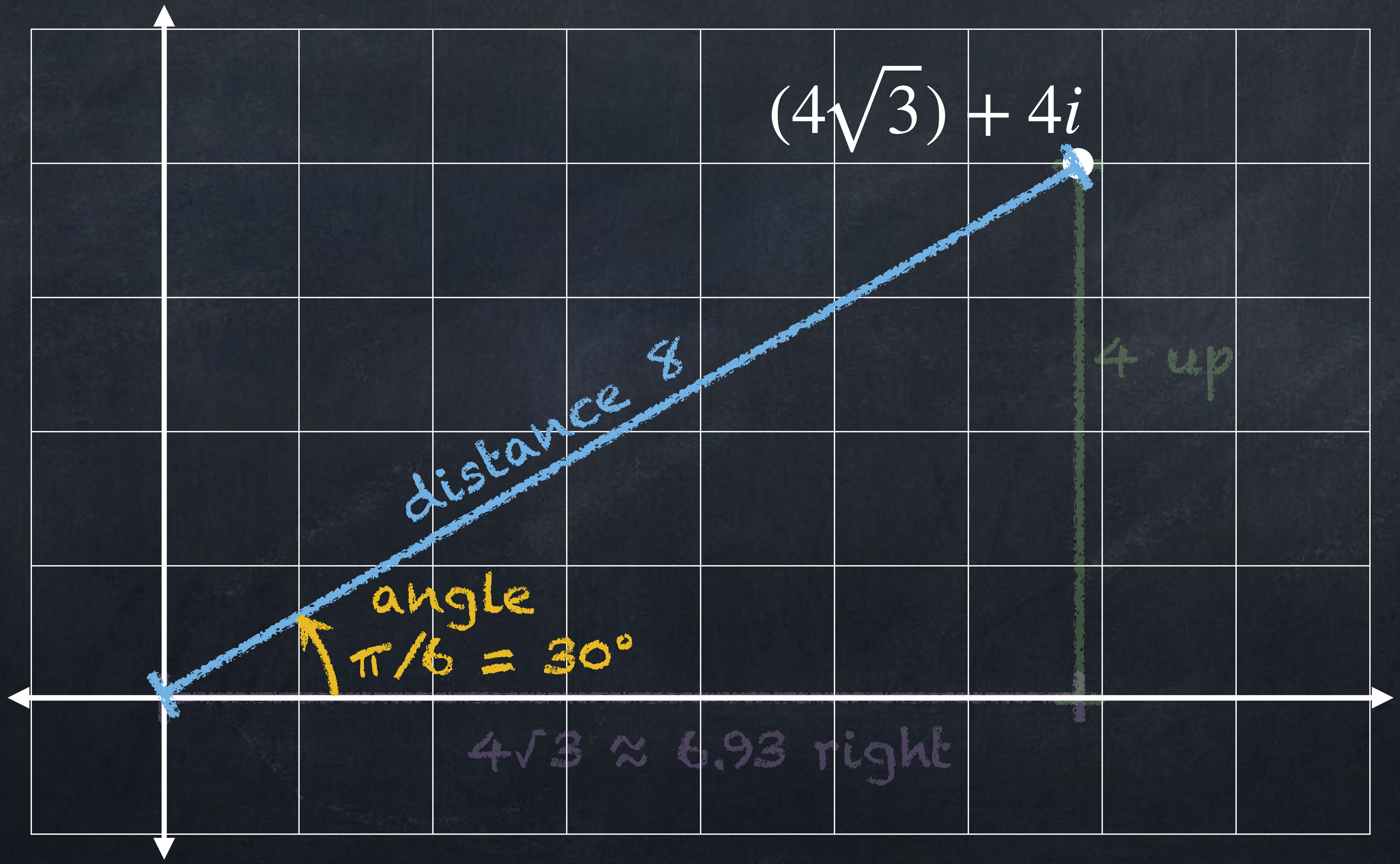

Magnitude and argument

The **magnitude** (also called **modulus** or **Euclidean norm**) of a complex number is its distance from 0.

- We write $|z|$ for the modulus of a complex number z.
- Formula: $|a + bi| = \sqrt{a^2 + b^2}$ if a and b are real

The **argument** of a complex number is the angle between the positive real axis and the line from 0 to that complex number.

- We write $\arg(z)$ for the modulus of a complex number z.

Example: $\left|4\sqrt{3} + 4i\right| = 8$ and $\arg\left(4\sqrt{3} + 4i\right) = 30° = \frac{\pi}{6}$.

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$$(4\sqrt{3}) + 4i = 8\cos(\tfrac{\pi}{6}) + 8\sin(\tfrac{\pi}{6})\, i$$

If we can write a complex number $z = a + bi$ as

$$z = r\cos(\theta) + r\sin(\theta)\, i$$

then we get $|\, z\,| = r$ and $\arg(z) = \theta$ easily.

Polar notation #1

$r\cos(\theta) + r\sin(\theta)\,i$ is a useful way to write a complex number z (it makes it easy to find $|z|$ and $\arg z$), and there is a shorter way to write it:

$\mathrm{cis}(\theta)$ means $\cos(\theta) + i\sin(\theta)$

and so

$r\,\mathrm{cis}(\theta) = r\cos(\theta) + r\sin(\theta)\,i$.

There is another, *even shorter* way to write this same number. Before showing Polar notation #2, let's look at z^2 in this format.

For

$$z = 8\cos(\pi/6) + 8\sin(\pi/6)\, i,$$

calculate z^2.

$$
\begin{aligned}
z^2 &= ((4\sqrt{3}) + 4i)^2 \\
&= (4\sqrt{3})^2 + 2(4\sqrt{3})(4i) + (4i)^2 \\
&= (4^2 \cdot 3 - 4^2) + (2 \cdot 4 \cdot 4 \cdot \sqrt{3})i \\
&= 32 + (32\sqrt{3})i \\
&= 32(1 + \sqrt{3})i \\
&= 64\left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)
\end{aligned}
$$

$$z^2 = 64\cos(\pi/3) + 64\sin(\pi/3)\, i$$

de Moivre's Formula (version 1)

For any integer n,

$$(r\cos(\theta) + r\sin(\theta)\, i)^n = r^n \cos(n\theta) + r^n \sin(n\theta) i.$$

What does this formula look like if we use "cis"?

de Moivre's Formula (version 2)

For any integer n, $\left(r\,\mathrm{cis}(\theta)\right)^n = r^n\,\mathrm{cis}(n\theta)$.

Polar notation #2

We will write

$$re^{i\theta}$$

instead of $r\,\mathrm{cis}(\theta)$ or the long version $r\cos(\theta) + r\sin(\theta)\,i$.

We have many ways to write numbers: $6\frac{1}{2} = \frac{13}{2} = 6.5$.

Now $1+\sqrt{3}i = 2\cos(\frac{\pi}{3}) + 2i\sin(\frac{\pi}{3}) = 2\,\mathrm{cis}(\frac{\pi}{3}) = 2e^{\frac{\pi}{3}i}$.

de Moivre's Formula (version 2)

For any integer n, $\left(r\,\mathrm{cis}(\theta)\right)^n = r^n\,\mathrm{cis}(n\theta)$.

Polar notation #2

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de Moivre's Formula (version 2)

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de Moivre's Formula (version 3)

For any integer n, $\left(r\,e^{\theta i}\right)^n = r^n\,e^{(n\theta)i}$.

basic algebra rules

Polar notation #2

Example: Write $5 + 5i$ in polar form $\underline{5\sqrt{2}}\, e^{(\frac{\pi}{4} i)}$.

- Hint: find $|5 + 5i|$ and $\arg(5 + 5i)$ first.

Multiplication

Multiplying $\left(\frac{-7}{2}+\frac{7\sqrt{3}}{2}i\right)\left(2-2\sqrt{3}\,i\right)$ is possible, but it takes a lot of algebra work.

Multiplying $\left(7e^{\frac{2\pi}{3}i}\right)\left(4e^{\frac{-\pi}{3}i}\right)$ is very easy:

$$\left(7\cdot 4\right)e^{\left(\frac{2\pi}{3}i+\frac{-\pi}{3}i\right)} = 28e^{\left(\frac{\pi}{3}i\right)}.$$

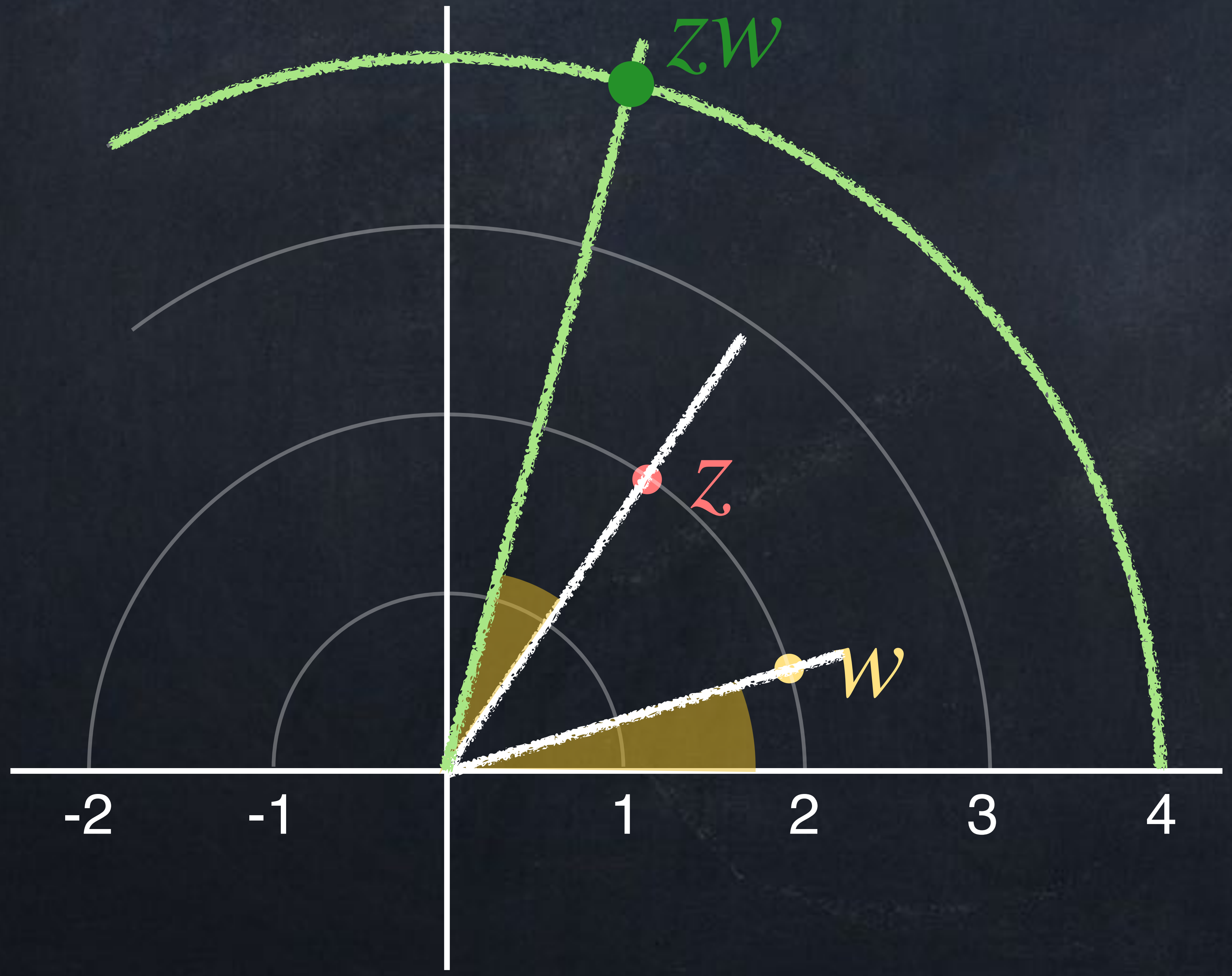
We can then expand this to $28\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)=14+14\sqrt{3}\,i$ if we need to.

Multiplication

In general, $$(re^{\theta i}) \cdot (se^{\phi i}) = (rs)e^{(\theta+\phi)i}.$$

What does this mean *visually*?

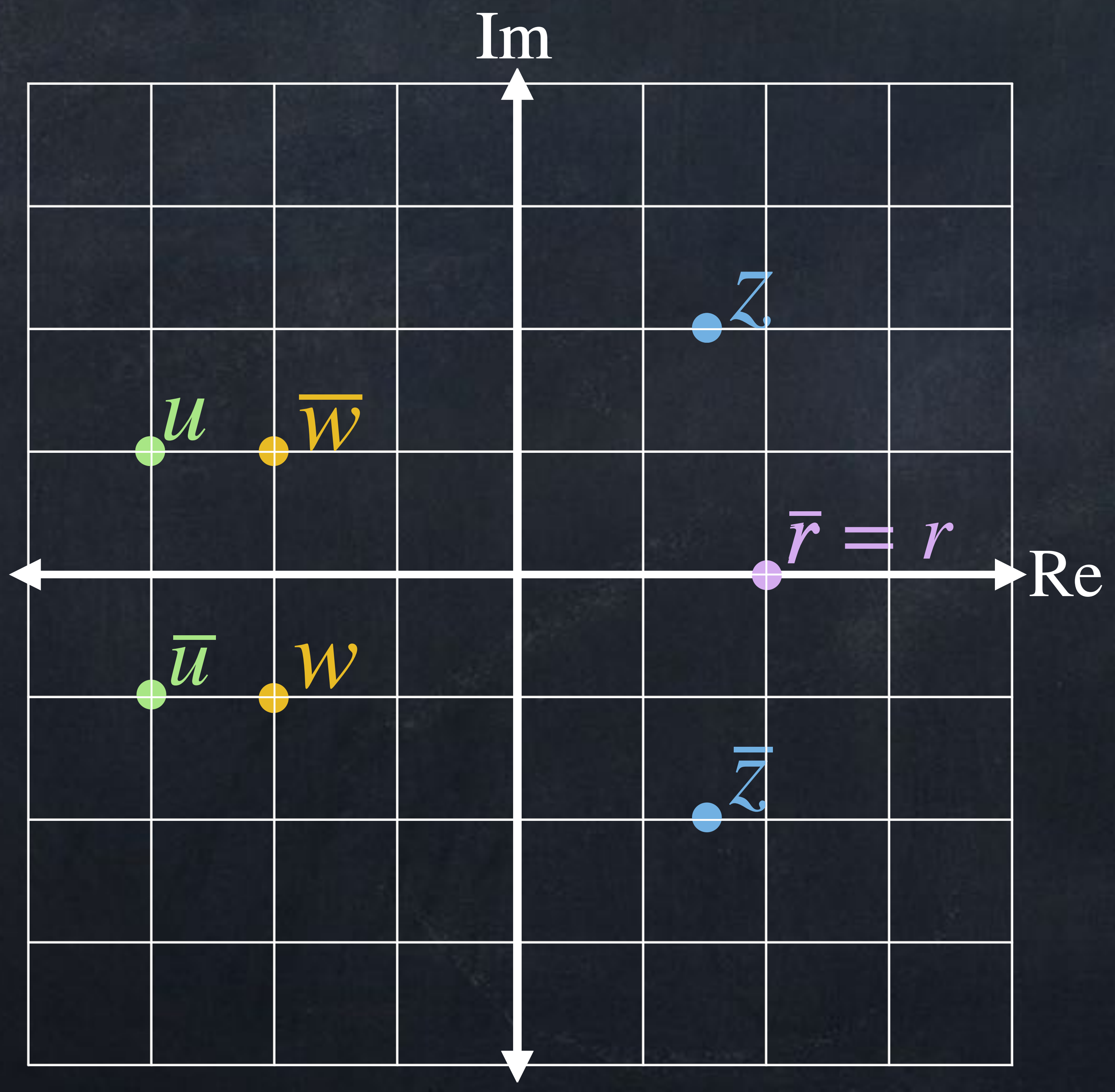
Let's draw a dot • at zw below.



Complex conjugate

The **complex conjugate** (or just **conjugate**) of a complex number z is the reflection of z across the real axis.

It is written $\overline{z}$ and spoken as "z bar".

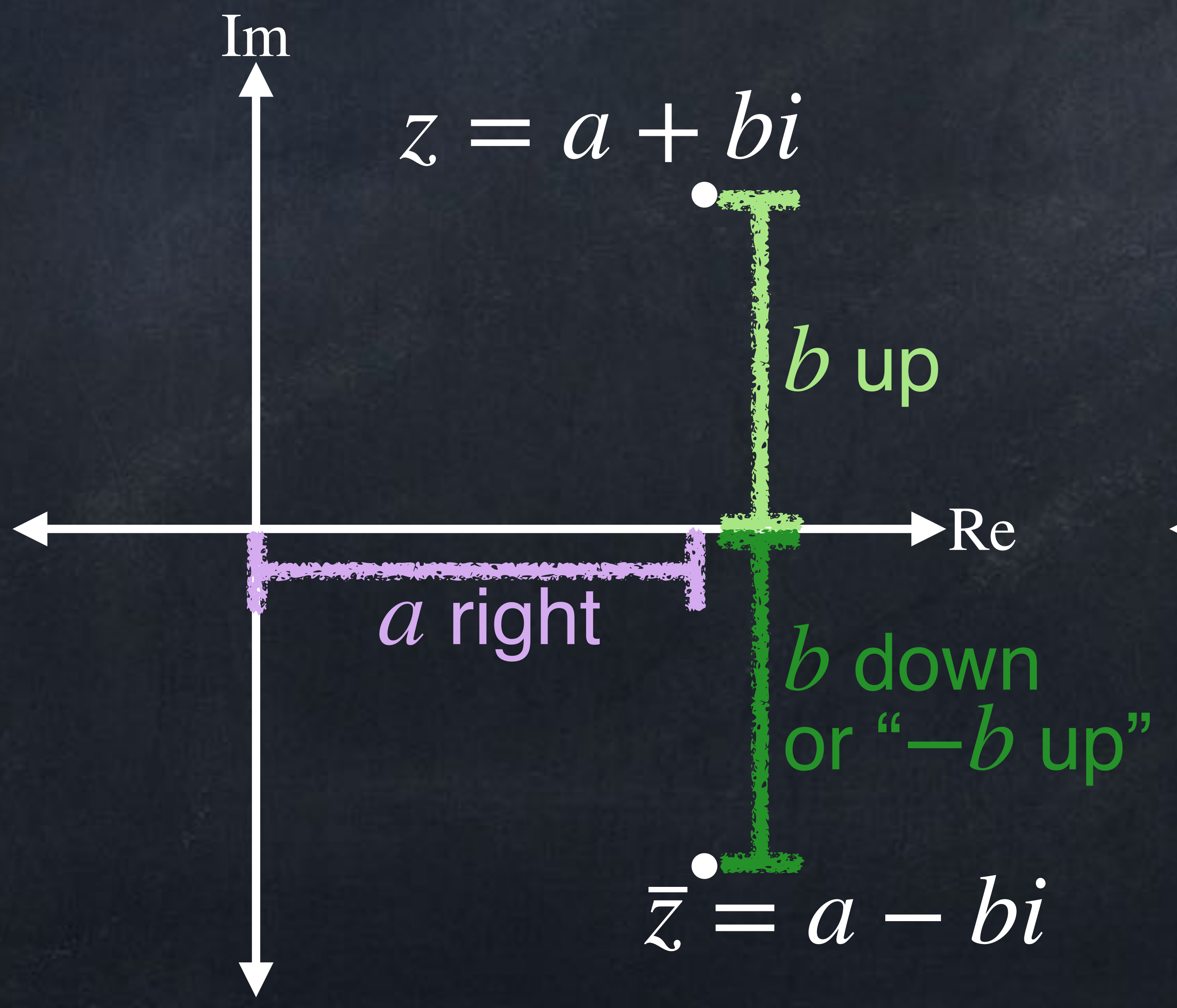


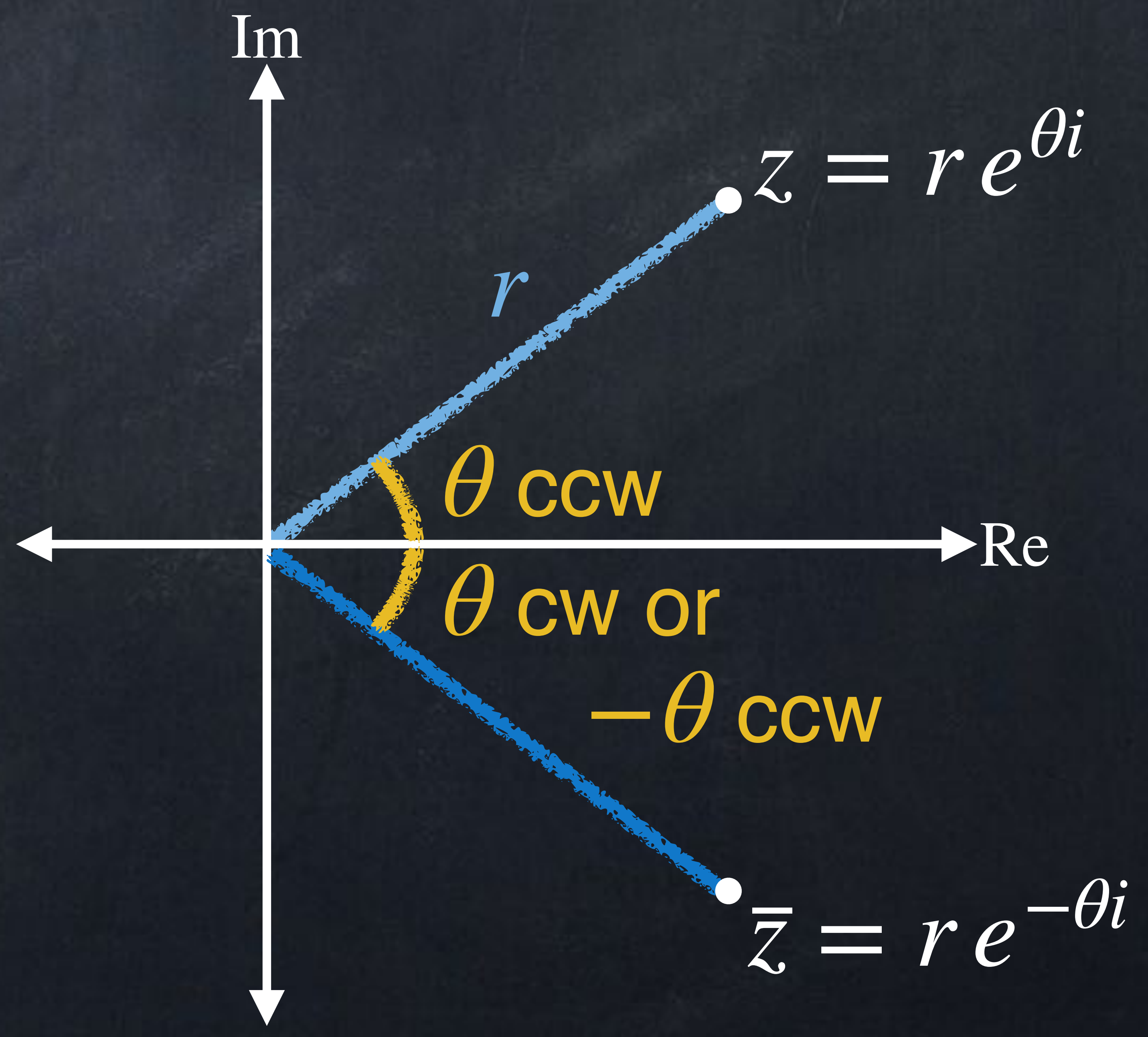
Complex conjugate

The **complex conjugate** (or just **conjugate**) of a complex number z is the reflection of z across the real axis.

What are rectangular and polar formulas for conjugates?

- $\overline{a+bi} = a - bi$
- $\overline{r\,e^{\theta i}} = r\,e^{-\theta i}$





How easy it is to do calculations in rectangular form and in polar form?

		$a+bi$	$re^{\phi i}$
real part	$\mathrm{Re}(z)$	😁	🙂
imaginary part	$\mathrm{Im}(z)$	😁	🙂
modulus	$\|z\|$	🙂	😁
argument	$\arg(z)$	😫	😁
conjugate	$\bar{z}$	😁	😁
sum/difference	$z \pm w$	😁	😫
product	zw	🙂	😁
quotient	z/w	😫	😁