Math 1688

Thursday, 21 October

Warm-up:
Go to theadamabrams.com/live

Roots of unity

What numbers z satisfy $z^2 = 1$?

Answer: 1, -1

What are all the complex numbers z that satisfy $z^4 = 1$?

Note that $z^2 = 1$ or $z^2 = -1$.

Answer: 1, -1, i, -i

Roots of unity

What are all the complex numbers z that satisfy $z^3 = 1$?

$$z^3 = (re^{\phi i})^3 = r^3 e^{(3\phi)i} = 1e^{(0°)i}$$

$r^3 = 1$ and $3\phi = 0°$ or $360°$ or $720°$ or $1080°$ ...

+360°, so same argument

$r = 1$ $\qquad \phi = 0°$ or $120°$ or $240°$ or $360°$ ...

same as 0°

$$z = 1e^{(0°)i} = 0$$
$$z = 1e^{(120°)i} = -1/2 + \sqrt{3}/2\ i$$
$$z = 1e^{(240°)i} = -1/2 - \sqrt{3}/2\ i$$

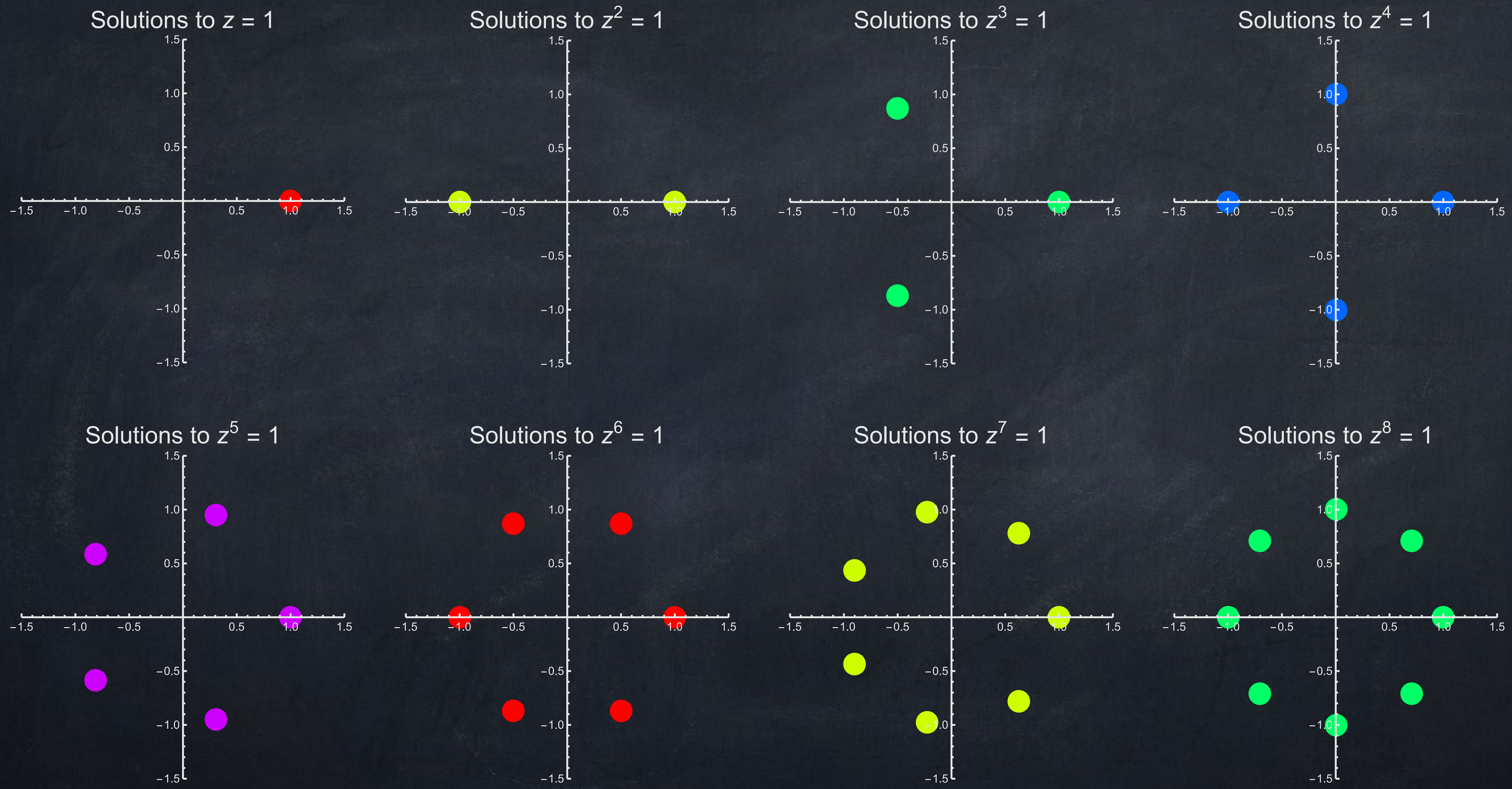
Roots of unity

For any natural number n, the solutions to $z^n = 1$ are exactly

- $z = e^{(2\pi/n)i}$
- $z = e^{2\cdot(2\pi/n)i}$
- $z = e^{3\cdot(2\pi/n)i}$

  $\vdots$
- $z = e^{(n-1)\cdot(2\pi/n)i}$
- $z = e^{n\cdot(2\pi/n)i} = e^{2\pi i} = 1.$

These are called the ***n*****th roots of unity**.


Solutions to $z = 1$
Solutions to $z^2 = 1$
Solutions to $z^3 = 1$
Solutions to $z^4 = 1$
Solutions to $z^5 = 1$
Solutions to $z^6 = 1$
Solutions to $z^7 = 1$
Solutions to $z^8 = 1$
−1.5 −1.0 −0.5 0.5 1.0 1.5

Real vs. complex

In some ways, real numbers are better.

- Physical measurements
- Ordered: always $x < y$ or $x \geq y$

In some ways, complex #s are better.

- n^{th} roots – always exactly n of them
- Rotation and trig functions
- Polynomials — …

Not true for real (example: $x^2+1 = 0$).

The Fundamental Theorem of Algebra (ver. 1)

For any non-constant polynomial $f(x)$, there is at least one complex solution to $f(x) = 0$.

Polynomials

A **polynomial** in the variable x is a function of real numbers that *can* be described by an expression of the form

$$😀 x^n + 🧐 x^{n-1} + \cdots + 😂 x^2 + 🥺 x + 😐,$$

where $n \geq 0$ is an integer and the emoji are real or complex numbers (called the **coefficients**).

A **real polynomial** is one where every coefficient is a real number.

A **complex polynomial** is one where every coefficient is complex.

- Real numbers are complex numbers ($a + 0i$), so every real polynomial is also a complex polynomial.

Polynomials

Examples of polynomials:

- $5x^3 - 27x + \frac{3}{2}$
- $\sqrt{82}x^5 - 9x$
- $(x-1)^3$ ⟵ This can be written as x^3+3x^2+3x+1, so it is a polynomial.
- 12
- $ax+b$ if the variable is x
- $7t^2 - 8t + 1$ if the variable is t

Examples that are *not* polynomials:

- x^{-3}
- $5x^2 + 3 + x^{-1}$
- $\sin(x)$

Roots or zeros

The number c is a **zero** of the polynomial f if $f(c) = 0$. A zero of a polynomial is also called a **root** of the polynomial.

Sometimes we are interested in particular types of numbers as zeros.

- Example: $2x^6 - 3x^5 - 21x^4 + 56x^3 - 26x^2 - 245x + 525$ has
  - Integer root: -3
  - Rational roots: -3 and $\frac{5}{2}$
  - Real roots: $-3, \frac{5}{2}, \sqrt{7},$ and $-\sqrt{7}$
  - Complex roots: $-3, \frac{5}{2}, \sqrt{7}, -\sqrt{7}, 1+2i,$ and $1-2i$

Roots or zeros

The number c is a **zero** of the polynomial f if $f(c) = 0$. A zero of a polynomial is also called a **root** of the polynomial.

We often use the variable z when we care about complex roots.
For example,

- "What are the zeros of $x^2 + 1$?"

  Depending who you ask, the answer could be either "i and $-i$" or "none" (there are no zeros).

- "What are the zeros of $z^2 + 1$?"

  Answer: i and $-i$.

Roots or zeros

The number c is a **zero** of the polynomial f if $f(c) = 0$. A zero of a polynomial is also called a **root** of the polynomial.

The Fundamental Theorem of Algebra (ver. 1)

Every non-constant complex polynomial has at least one root.

Finding roots by hand

Example: Find all roots of $z^2 + (1+i)z + i$.

Degree

The **degree** of a polynomial is the highest power of the variable that appears in the polynomial. We write $\deg(f)$ for the degree of $f(x)$.

- Degree 0 example: 9 "constant"
- Degree 1 example: $x + 2$ "linear"*
- Degree 2 example: $2x^2 - 5x - 12$ "quadratic"
- Degree 3 example: $-8x^3$ "cubic"
- Degree 4 example: $x^4 - 7x + 1$ "quartic"

$+ \quad - \quad \times \quad \div$

We can add two polynomials.

$$(4x^2 - 3x) + (x^3 + x^2 + 3x + 8) \quad = \quad x^3 + 5x^2 + 8$$

We can subtract two polynomials.

$$(4x^2 - 3x) - (x^3 + x^2 + 3x + 8) \quad = \quad -x^3 + 3x^2 - 6x - 8$$

We can multiply two polynomials.

$$(4x^2 - 3x)(x^3 + x^2 + 3x + 8) \quad = \quad 4x^5 + x^4 + 9x^3 + 23x^2 - 24x$$

We can try to divide two polynomials, but sometimes the result is not a polynomial (for example, 1/x is not a polynomial).

$+ \quad - \quad \times \quad \div$

Question: What can we say about $\deg(f+g)$ and $\deg(f \cdot g)$?

$$(4x^2 - 3x) + (x^3 + x^2 + 3x + 8) = x^3 + 5x^2 + 8$$

$$(4x^2 - 3x) + (-4x^2 + 7) = -3x + 7$$

$\deg(f+g)$ is $\leq$ the maximum of $\deg(f)$ and $\deg(g)$.

$$(4x^2 - 3x)(x^3 + x^2 + 3x + 8) = 4x^5 + x^4 + 9x^3 + 23x^2 - 24x$$

$$= 4x^2(x^3+x^2+3x+8) + (-3x)(x^3+x^2+3x+8)$$

$$= (4x^5 + \cdots) + (-3x^4 + \cdots)$$

$$x^a \cdot x^b = x^{a+b}$$

$\deg(f \cdot g) = \deg(f) + \deg(g)$ exactly.

Factoring

Natural numbers can be "factored" (re-written as a product of smaller numbers).

- Example: $198 = 6 \cdot 33$

If $a = b \cdot c$, we say that b is a **factor** of a.

A natural number other than 1 that cannot be factored is called a **prime** number. The first several primes are $2, 3, 5, 7, 11, 13, \ldots$

Factoring

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A natural number other than 1 that cannot be factored is called a **prime** number. The first several primes are $2, 3, 5, 7, 11, 13, \ldots$

We can uniquely factor a natural number as a product of primes.

- Example: $198 = 2 \cdot 3^2 \cdot 11$

(If we expand from naturals to integers, we might need to include -1.)

- Example: $-1625 = -1 \cdot 5^3 \cdot 13$

Factoring

Polynomials can also be factored. Examples:

- $x^2+8x = x(x+8)$
- $x^2+\frac{1}{2}x = x(x+\frac{1}{2})$
- $x^3-12x^2+41x-42 = (x^2-5x+6)(x-7)$
- $x^3-11x^2+34x-42 = (x^2-4x+6)(x-7)$

If $f(x) = g(x) \cdot h(x)$, we say that $g(x)$ is a **factor** of $f(x)$.

Factoring

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The Factor Theorem

If $(x - r)$ is a factor of the polynomial $f(x)$, then r is a zero of $f(x)$.
If r is a zero of the polynomial $f(x)$, then $(x - r)$ is a factor of $f(x)$.

This means that if we find one zero of $f(x)$—let's call this number r—then the other zeros of $f(x)$ will be zeros of $g(x) = \dfrac{f(x)}{x - r}$. Note that g has a lower degree than f.

Finding roots by hand

Example: Find all roots of $x^3 - 13x + 12$, given that 3 is a root.

Slow method: algebra rules

$$(x^3 - 13x + 12) = (x-3)(ax^2 + bx + c) \text{ for some } a,b,c$$
$$= ax^3 + bx^2 + cx - 3ax^2 - 3bx - 3c$$
$$= ax^3 + (b-3a)x^2 + (c-3b)x + (-3c)$$

For $ax^3 + \cdots$ to equal equal $1x^3 + 0x^2 - \cdots$, we must have

$$a = 1, \quad b-3a = 0, \quad c-3b = -13, \quad -3c = 12.$$

This leads to $a=1$, $b=3$, $c=-4$, so

$$(x^3 - 13x + 12) = (x-3)(x^2 + 3x - 4).$$

The roots of x^2+3x-4 are 1 and -4, so the roots of $x^3 - 13x + 12$ are **1, −4, and 3.**

Factoring

Polynomials can also be factored.

- If $f(x) = g(x) \cdot h(x)$, we say that $g(x)$ is a **factor** of $f(x)$.

A polynomial that *cannot* be factored as a product of non-constant polynomials is called **irreducible**.

Note $2x+10$ is irreducible.
$2x+10 = 2(x+5)$ is like $-23 = (-1)(23)$.

- *Question:* How can you tell when you have only irreducible factors?

$$x^3 - 12x^2 + 41x - 42 = (x^2 - 5x + 6)(x - 7) = (x-2)(x-3)(x-7)$$

$$x^3 - 11x^2 + 34x - 42 = (x^2 - 4x + 6)(x - 7)$$

Irreducible factors

Question: How can you tell when a polynomial is irreducible?

- Any linear polynomial must be irreducible.

  Proof: If $f(x) = ax + b$ is equal to some product $g(x)h(x)$, then $\deg(f) = \deg(g) + \deg(h)$, but we know $\deg(f) = 1$, and if g and h are not constant then we know $\deg(g) \geq 1,\ \deg(h) \geq 1$.

- What about quadratics?

Irreducible factors

Question: How can you tell when a polynomial is irreducible?

- Any linear polynomial must be irreducible.
- What about quadratics?
  - The roots of ax^2+bx+c are $\frac{-b\pm\sqrt{D}}{2a}$, where $D = b^2 - 4ac$. The number D is called the **discriminant**.
  - With complex numberss we can always factor quadratics:

$$ax^2+bx+c = \left(x-\frac{-b+\sqrt{D}}{2a}\right)\left(x-\frac{-b-\sqrt{D}}{2a}\right).$$

But for real numbers we need $D \geq 0$ in order to use $\sqrt{D}$.

Irreducible factors

Question: How can you tell when a polynomial is irreducible?

Answer: It depends on whether you allow complex numbers.

- An irreducible real polynomial is *either* linear *or* quadratic with negative discriminant.
- An irreducible complex polynomial is linear.

Example:

$$x^4 + x^3 - 21x^2 + 9x - 270 \quad = \quad (x-5)(x+6)(x^2+9)$$

is completely factored as a real polynomial. But if we allow complex numbers then

$$x^4 + x^3 - 21x^2 + 9x - 270 \quad = \quad (x-5)(x+6)(x+3i)(x-3i)$$

Irreducible factors

The Fundamental Theorem of Algebra (ver. 2)

A complex polynomial of degree n can be factored into exactly n irreducible (linear) factors.

Example:

$$x^4 + x^3 - 21x^2 + 9x - 270 \quad = \quad (x-5)(x+6)(x^2+9)$$

but

$$z^4 + z^3 - 21z^2 + 9z - 270 \quad = \quad (z-5)(z+6)(z+3i)(z-3i)$$

Irreducible factors

The Fundamental Theorem of Algebra (ver. 2)

A complex polynomial of degree n can be factored into exactly n irreducible (linear) factors.

- $f(z)$ with degree n can always be factored as

$$f(z) = a(z - c_1)(z - c_2)(z - c_3)\cdots(z - c_n)$$

where a is a constant and $c_1, \ldots, c_n$ are (possibly repeated) roots.

Irreducible factors

The Fundamental Theorem of Algebra (ver. 2)

A complex polynomial of degree n can be factored into exactly n irreducible (linear) factors.

How many prime factors does 34024771 have? 🤷

How many irreducible real factors does $x^7 + \sqrt{2}\,x^5 - x^4 + \frac{2}{9}x - 8$ have? 🤷

How many irreducible complex factors does $9z^{12} - \frac{13}{5}z^7 - \sqrt[3]{\pi}\,z^4 + 57$ have? 12