Math 1688

Thursday, 28 October

Warm-up 1: Expand $(z - 4 - 2i)(z - 4 + 2i)$ to the form $\_\_z^2 + \_\_z + \_\_$.

Warm-up 2: What is the remainder of 14 divided by 6?

Last week

The Fundamental Theorem of Algebra (version 2.0)

Every non-constant polynomial of degree n is a product of exactly n linear complex polynomials.

The Factor Theorem

$(x - c)$ is a factor of the polynomial $f(x)$ if and only if $f(c) = 0$.

Real coefficients

If $(az^2 + bz + c)$ is a factor of $f(z)$ then $\dfrac{-b}{2a} + \dfrac{\sqrt{D}}{2a}$ and $\dfrac{-b}{2a} - \dfrac{\sqrt{D}}{2a}$ are zeros of $f(z)$.

If, also, a and b and c are real numbers, then D is also a real number.

- If $D \geq 0$, the two zeros above are real numbers.
- If $D < 0$, then $\sqrt{D} = i\sqrt{|D|}$ and so those zeros are $\left(\dfrac{-b}{2a}\right) + \left(\dfrac{\sqrt{|D|}}{2a}\right)i$ and $\left(\dfrac{-b}{2a}\right) - \left(\dfrac{\sqrt{|D|}}{2a}\right)i$.

Real coefficients

The Conjugate Pairs Theorem

If $a + bi$ is a root of a real polynomial, then $a - bi$ is also a root of that polynomial.

Example: $4 + 2i$ is one of the zeros of $f(z) = z^3 - 11z^2 + 44z - 60$.
Knowing this, find all the zeros of f.

Division

- Given two polynomials f and g, we know $f+g, f-g, f\times g$ are always polynomials. What about $f\div g$?
- A fraction with polynomials is called a **rational function**.
  - Examples: $\dfrac{5x}{2x-7}$ $\dfrac{9x^3-8x^2}{2x+1}$ $\dfrac{57}{x^{10}+4}$
  - *Sometimes* you can simplify to polynomial,

$$\frac{x^2-9}{x+3} = \frac{(x+3)(x-3)}{x+3} \rightsquigarrow x-3,$$

but *sometimes* you cannot: $\dfrac{1}{x} = x^{-1}$ is definitely not a polynomial.

Division

- Integers
  - Sometimes when you divide two integers you get an integer.

$$\frac{12}{3} = 4$$

  - Sometimes you don't.

$$\frac{26}{7} = 3 + \frac{5}{7}$$

    - Then you have a remainder.

$$3 \text{ remainder } 5$$

- Polynomials
  - Sometimes a rational function simplifies to a polynomial.

$$\frac{2x^2 - 18}{x + 3} \rightsquigarrow 2x - 6$$

  - Sometimes it does not.

$$\frac{2x^2 - 18}{x} = 2x + \frac{-18}{x}$$

    - Then you have a remainder.

$$2x \text{ remainder } (-18)$$

Division

$$\frac{a}{b} = q + \frac{r}{b}$$

$$\frac{f(x)}{g(x)} = Q(x) + \frac{R(x)}{g(x)}$$

Division of Integers

Given integers a and b with $b > 0$, there exist *unique* integers q and r such that

$$a = b \cdot q + r$$

and $0 \leq r < b$.

Division of Polynomials

Given any polynomials $f(x)$ and $g(x)$, there exist *unique* polynomials $Q(x)$ and $R(x)$ such that

$$f(x) = g(x) \cdot Q(x) + R(x)$$

and $R(x) = 0$ or $\deg(R) < \deg(g)$.

- Q is called the quotient and R is called the remainder.

Division example

When x^3+2x^2+5 is divided by x^2-3,
the quotient is $x+2$
and the remainder is $3x+11$.

- This means $x^3+2x^2+5 = (x^2-3)(x+2) + (3x+11)$.
  - It's also true that $x^3+2x^2+5 = (x^2-3)(x+1) + (x^2+3x+8)$, but the degree of x^2+3x+8 is not less than $\deg(x^2-3) = 2$.
- You can also think of $\dfrac{x^3+2x^2+5}{x^2-3} = (x+2) + \dfrac{3x+11}{x^2-3}$.

How can we find these q and r ourselves?

Division example

Find the quotient and remainder when $x^5 + 6x^4 + 9x^3 - 3x^2 - 4$ is divided by $x^2 + 2x + 1$.

$$x^5+6x^4+9x^3-3x^2-4 = (x^2+2x+1)(x^3+4x^2+7)+(14x+3)$$

quotient remainder

$$-(x^5+2x^4+x^3)$$

$$4x^4+8x^3$$

$$-(4x^4+8x^3+4x^2)$$

$$-7x^2$$

$$-(-7x^2-14x-7)$$

$$14x+3$$

Note $x^5+2x^4+x^3$ is $(x^2+2x+1)(x^3)$, where x^3 is just part of the quotient.

Division by Linear

Ex: Find the remainder when $x^5 + 6x^4 + 9x^3 - 3x^2 - 4$ is divided by $x - 1$.

Division by Linear

The Remainder Theorem

The remainder when a polynomial $f(x)$ is divided by $(x - a)$ is the value of $f(a)$.

The Factor Theorem

$(x - a)$ is a factor of $f(x)$ if and only if $f(a) = 0$.

If (x-a) is a factor of f, then the remainder of f divided by (x-a) is zero. The Remainder Thm then tells us f(a) = 0, though in fact we already knew that from the Factor Theorem.

Division by Linear

- Find the remainder when $x^4 - 5x^2 + 4x + 2$ is divided by $x - 3$.

$$f(3) = 3^4 - 5(3^2) + 4(3) + 2 = 50$$

- Find the quotient when $x^4 - 5x^2 + 4x + 2$ is divided by $x - 3$.
  - We want $x^4 - 5x^2 + 4x + 2 = (x - 3)Q(x) + 50$.
  - The fast way to do this is with "synthetic division".

	1	0	-5	4	2
3		3	9	12	48
	1	3	4	16	50

$$x^3 + 3x^2 + 4x + 16$$

Repeated roots

- A number r is a **root** of f (we can also say r is a **zero** of f) if

$$f(r) = 0.$$

  - This implies that $(x - r)$ is a factor of $f(x)$.
  - That means $f(x) = (x - r)g(x)$ for some polynomial g.

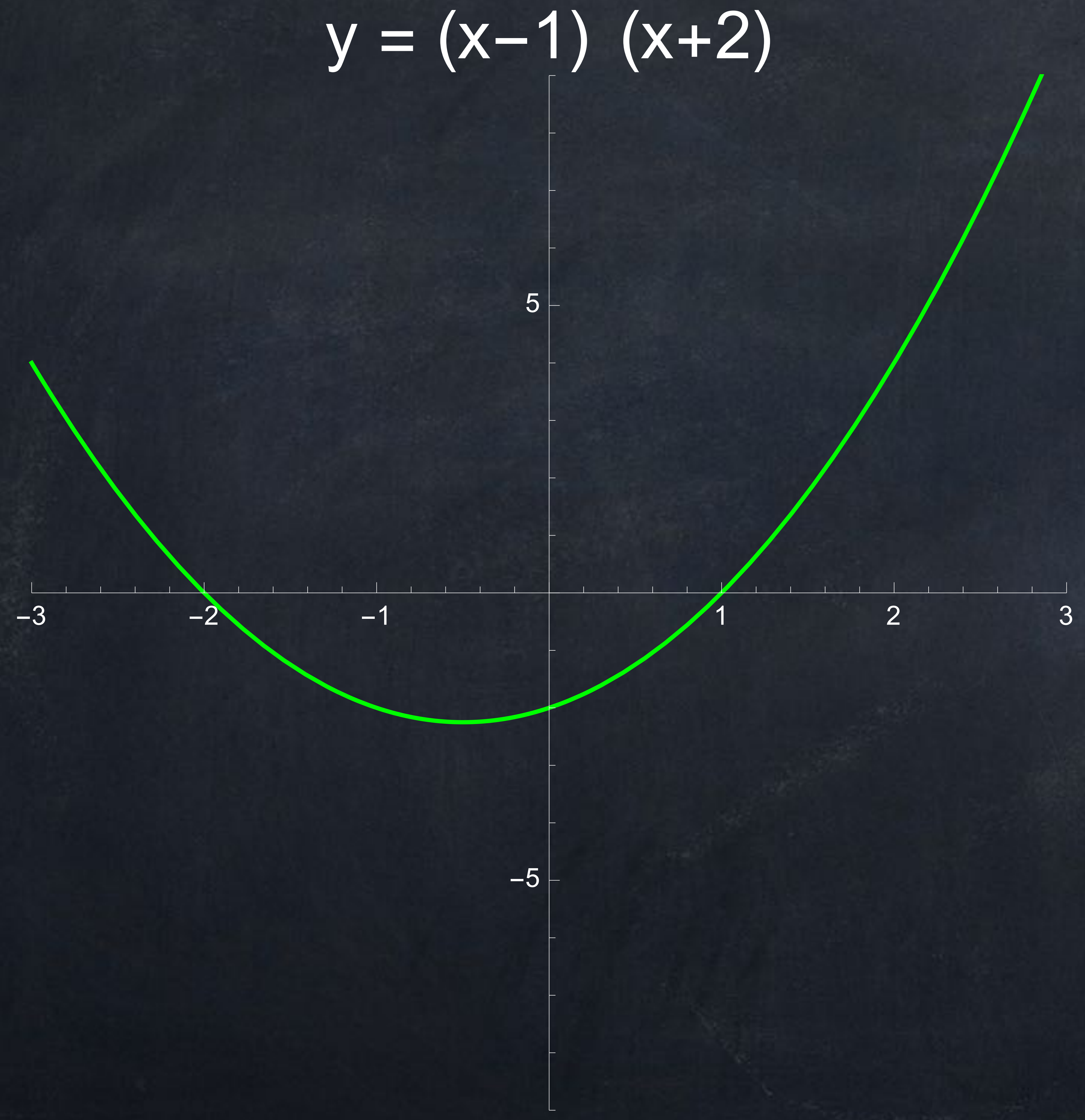
- The **multiplicity** of the root r is the highest number k for which

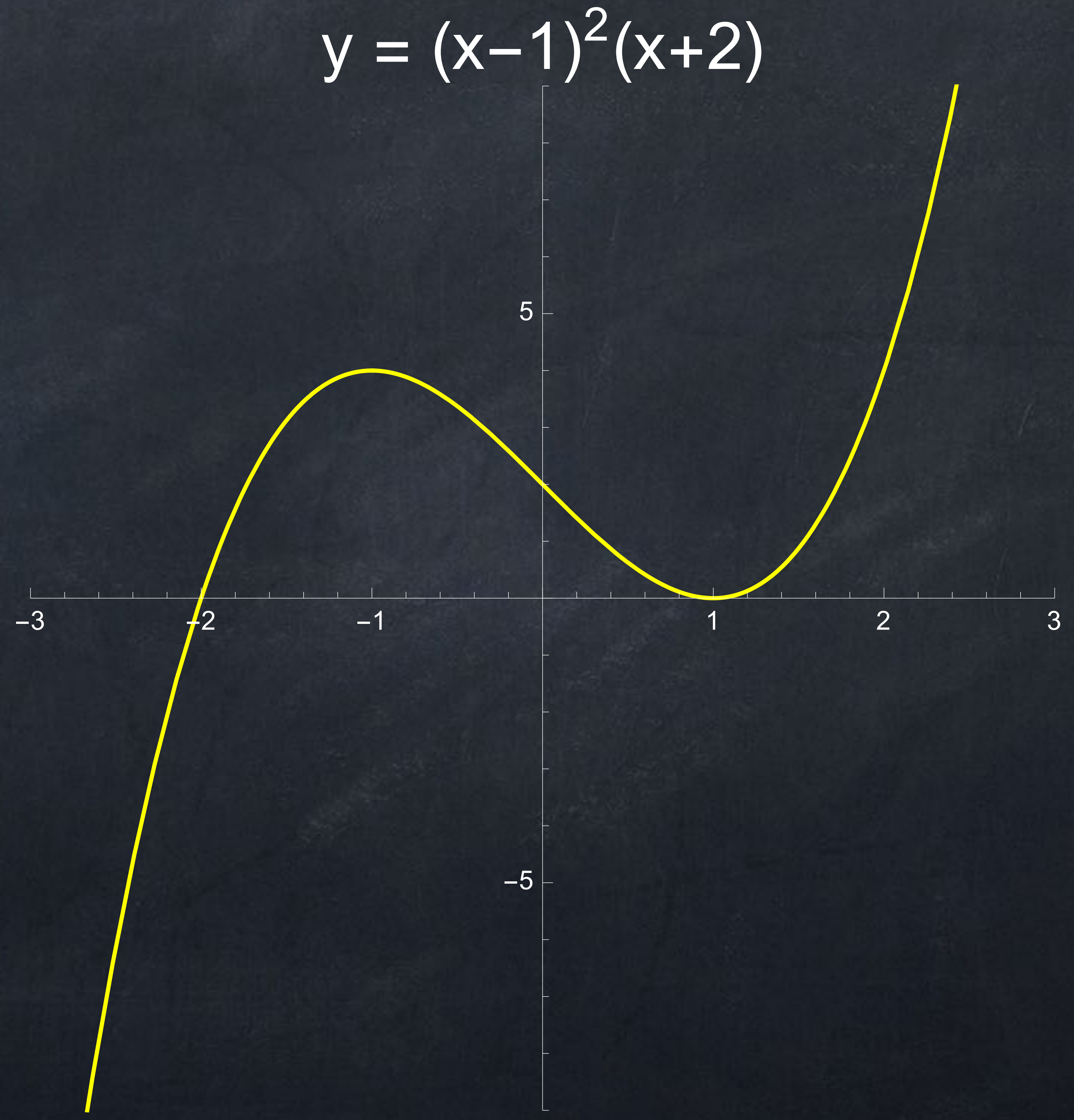
$$f(x) = (x - r)^k g(x)$$

for some polynomial g.

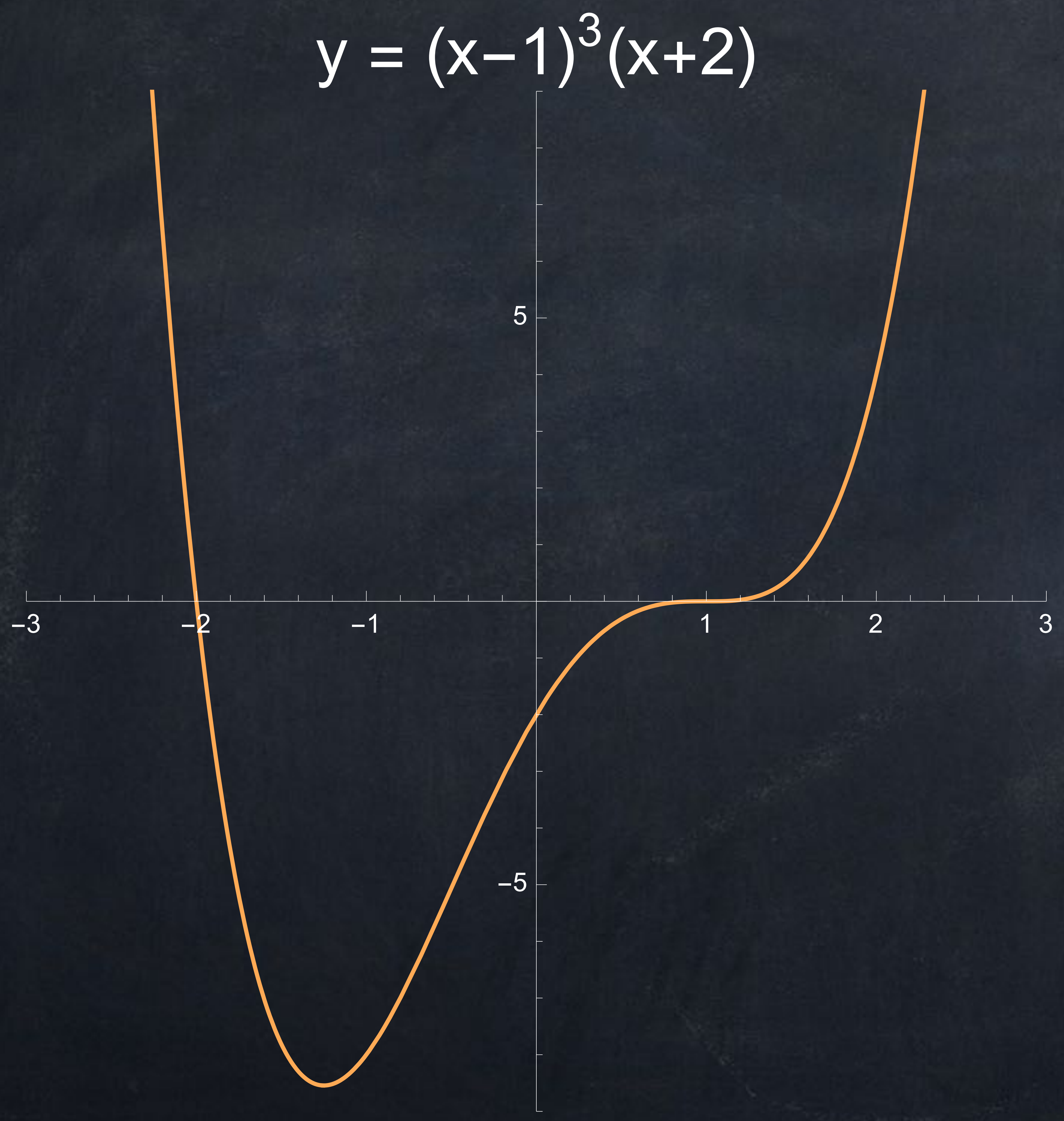
  - If $k > 1$ we say that r is a **repeated root** of f.

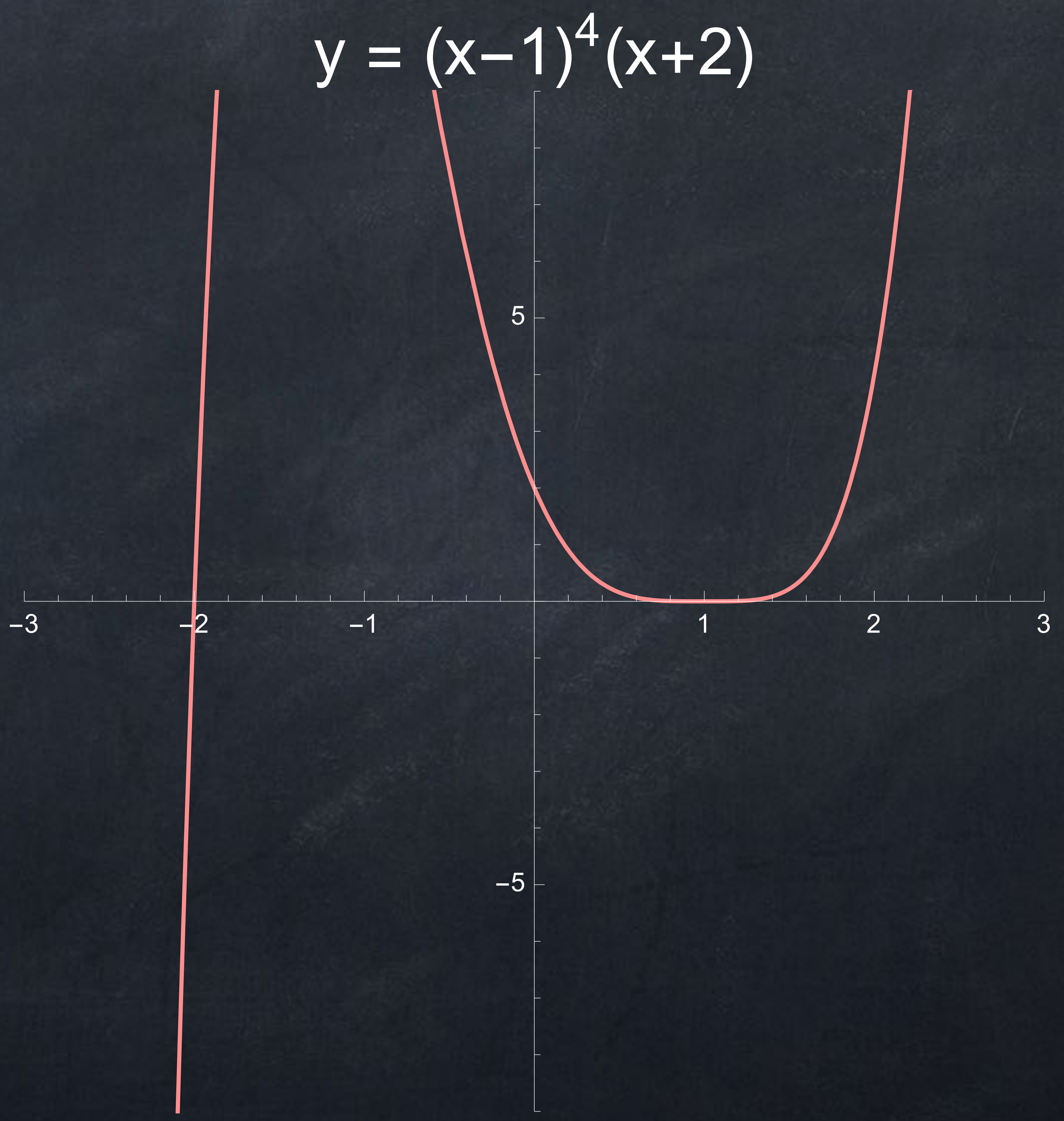
Roots of a polynomial





Roots of a polynomial





Using synthetic division

- The polynomial $x^4 + 4x^3 - 18x^2 + 20x - 7$ has $x = 1$ as a root. What is the multiplicity of this root?

	1	4	-18	20	-7
1		1	5	-13	7
	1	5	-13	7	0
1		1	6	-7	
	1	6	-7	0	
1		1	7		
	1	7	0		
1		1			
	1	8			

Answer: 3

8 ← not divisible by x-1 anymore → stop

Roots of a polynomial

- Remember that $f(z)$ with degree n can always be factored as
$$f(z) = a(z - c_1)(z - c_2)(z - c_3)\cdots(z - c_n)$$
where a is a constant and $c_1, \ldots, c_n$ are (possibly repeated) roots.

- If we list multiplicities, we have
$$f(z) = a(z - r_1)^{m_1}(z - r_2)^{m_2}\cdots(z - r_k)^{m_k}$$
for the k distinct roots, where m_i is the multiplicity of the root r_i.

Repeated roots

The Fundamental Theorem of Algebra (ver. 3)

A polynomial of degree n has exactly n complex roots, counted with multiplicities.

- Example: $f(z) = z^7+11z^6+41z^5+43z^4-69z^3-135z^2+27z+81.$
  - The only numbers for which $f(z) = 0$ are -3, -1, and 1.
  - Since $f(z) = (z+3)^4(z+1)(z-1)^2$, we can think of the zeros of f as

$$-3,\ -3,\ -3,\ -3,\ -1,\ 1,\ 1.$$

Repeated roots

How many roots does $97344x^2 - 327600x + 275625$ have?

- Distinct real roots: Is $D = b^2 - 4ac$ positive, zero, or negative? 🤔
  ↓ two roots ↓ one root ↓ no roots
- Distinct complex roots: Is $D = b^2 - 4ac$ is zero or not? 🤔
  ↓ one root ↓ two roots
- Complex roots with multiplicities: 2. 😀