Math 1688

Thursday, 4 November

Warm-up: Re-write $\dfrac{3}{x+1} + \dfrac{1}{x}$ as one fraction.

theadamabrams.com/live

The “best” way to write a polynomial depends on your goal.

- $x^2 + 3x + 2$ is standard form. It is good for testing whether two polynomials are exactly equal.
- $(x+3)x+2$ is good for plugging in x-values (only one multiplication).
- $\left(x+\frac{3}{2}\right)^2 - \frac{1}{4}$ is good for graphing (vertex is at $\left(\frac{-3}{2}, \frac{-1}{4}\right)$).
- $(x+1)(x+2)$ is good for finding zeros.

The "best" way to write a number also depends on your goal.

Fractions

How can we describe 35 divided by 12?

- 2 remainder 11
- 2.916666...
- $\frac{35}{12}$
- $2 + \frac{11}{12}$
- $2 + \frac{2}{3} + \frac{1}{4}$

$\frac{\text{35 apples}}{\text{12 people}} =$  per person

$\frac{\text{35 apples}}{\text{12 people}} =$  per person

Fractions

$$\frac{2}{3} + \frac{1}{4} = \frac{2 \times 4}{3 \times 4} + \frac{1 \times 3}{4 \times 3} = \frac{8}{12} + \frac{3}{12} = \frac{11}{12}$$

The exact same process can be used with problems like the warm-up.

Warm-up: Re-write $\dfrac{3}{x+1} + \dfrac{1}{x}$ as one fraction.

Rational functions

A **rational function** is a fraction with polynomials.

The “best” way to write a rational function depends on your goal.

- $\dfrac{x^2 - 6x + 5}{x^2 + 3x + 2}$ is standard form.
- $\dfrac{(x-5)(x-1)}{(x+2)(x+1)}$ is good for graphing (numerator tells us the zeros and denominator tells us the vertical asymptotes).
- $1 + \dfrac{-21}{x+2} + \dfrac{12}{x+1}$ is good for derivatives and integrals (Analysis).

Partial fractions

We call $\dfrac{h(x)}{g(x)}$ a **partial fraction** if $g(x) = (P(x))^n$ for some irreducible polynomial $P(x)$ and integer $n \geq 1$ with, $\deg(h) < \deg(P)$.

Examples:

- $\dfrac{9}{3x-5}$
- $\dfrac{9}{(3x-5)^2}$
- $\dfrac{4x+2}{x^2+9}$

Same *idea* as $\dfrac{2}{3}$ and $\dfrac{1}{4}$.

prime

power of a prime

An **irreducible** polynomial *can't* be factored as a product of non-constant polynomials.

- Complex → only linear
- Real → linear or quadratic with $D < 0$

Partial Fraction Decomposition

Any rational function $\frac{f(x)}{g(x)}$ can be written as a sum of a polynomial and some number (possibly 0) of partial fractions.

How?

- First write $\dfrac{f(x)}{g(x)} = Q(x) + \dfrac{R(x)}{g(x)}$, where Q and R are the quotient and remainder from last week. Recall $\deg(R) < \deg(g)$.
  - Note: If $\deg(f) < \deg(g)$ then $Q(x) = 0$.
- Then we need to write $\frac{R(x)}{g(x)}$ as a sum of partial fractions.

★ **Each irreducible factor of $g(x)$ will be used as the denominator of a partial fraction.**

Example 1: Write $\dfrac{f(x)}{g(x)} = \dfrac{x^4 + x^3 - 4x^2 + 1}{x^2 + x}$ as a sum of a polynomial and some partial fractions.

First, we need Q and R.

quotient remainder

$$x^4+x^3-4x^2+0x+1 = (x^2+x)(x^2-4) + (4x+1)$$

So $\dfrac{x^4+x^3-4x^2+1}{x^2+x} = (x^2-4) + \dfrac{4x+1}{x^2+x}$.

Now we need to split $\dfrac{4x+1}{x^2+x}$ into partial fractions.

From the warm-up, we know $\dfrac{4x+1}{x^2+x} = \dfrac{3}{x+1} + \dfrac{1}{x}$.

Answer: $\boxed{\dfrac{x^4+x^3-4x^2+1}{x^2+x} = (x^2-4) + \dfrac{3}{x+1} + \dfrac{1}{x}}$

Example 2: Write $\dfrac{13x+9}{x^2+3x-10}$ as a sum of partial fractions.

Since $x^2+3x-10=(x-2)(x+5)$, we are looking for

$$\frac{13x+9}{(x-2)(x+5)} = \frac{A}{x-2} + \frac{B}{x+5}$$

$$\frac{13x+9}{(x-2)(x+5)} = \frac{A}{(x-2)}\frac{(x+5)}{(x+5)} + \frac{B}{(x+5)}\frac{(x-2)}{(x-2)}$$

$$13x+9 = A(x+5) + B(x-2)$$

Example 2: Write $\dfrac{13x+9}{x^2+3x-10}$ as a sum of partial fractions.

Since $x^2+3x-10 = (x-2)(x+5)$, we are looking for

$$\frac{13x+9}{(x-2)(x+5)} = \frac{A}{x-2} + \frac{B}{x+5}$$

$$13x+9 = A(x+5) + B(x-2)$$

$$13x+9 = Ax + 5A + Bx - 2B$$

$$13x+9 = (A+B)x + (5A-2B)$$

$$\begin{cases} A + B = 13 \\ 5A - 2B = 9 \end{cases} \longrightarrow A = 5 \text{ and } B = 8 \longrightarrow \boxed{\frac{5}{x-2} + \frac{8}{x+5}}$$

Answer

More difficult partial frac.

If $g(x) = (x-a)(x-b)\cdots$ with *distinct* linear factors, then writing $\frac{f(x)}{g(x)}$ as a sum of partial fractions is just like our previous example.

$$\frac{f(x)}{2x^3+7x^2-53x-28} = \frac{f(x)}{(x-4)(2x+1)(x+7)}$$

$$= \frac{A}{x-4} + \frac{B}{2x+1} + \frac{C}{x+7}$$

$$\text{for some } A, B, C$$

More difficult partial frac.

"Simple" examples: $$\frac{f(x)}{(ax+b)(cx+d)\cdots} = \frac{A}{ax+b} + \frac{B}{cx+d} + \cdots$$

If g has an irreducible quadratic (degree 2) factor, we need a linear numerator (degree 1) for that fraction:

- $$\frac{f(x)}{(ax^2+bx+c)\cdots} = \frac{Ax+B}{ax^2+bx+c} + \cdots$$

If g has repeated zeros, we need a partial fr. for each power:

- $$\frac{f(x)}{(x-r)^3\cdots} = \frac{A}{x-r} + \frac{B}{(x-r)^2} + \frac{C}{(x-r)^3} + \cdots$$

Partial fraction examples:

- $\dfrac{9}{3x-5}$
- $\dfrac{4x+2}{x^2+9}$
- $\dfrac{9}{(3x-5)^2}$

Example 3: Write $\dfrac{f}{g} = \dfrac{x^2 + 3x - 4}{x^3 - 6x^2 + 4x - 24}$ as a sum of partial fractions.

Hint: $g(6) = 0$.

$x^3 - 6x^2 + 4x - 24 = (x-6)(x^2+4)$, so we are looking for

$$\frac{x^2+3x-4}{(x-6)(x^2+4)} = \frac{A}{x-6} + \frac{Bx+C}{x^2+4}$$

$$x^2+3x-4 = A(x^2+4) + (Bx+C)(x-6)$$

$$x^2+3x-4 = Ax^2 + 4A + Bx^2 + Cx - 6Bx - 6C$$

$$x^2+3x-4 = (A+B)x^2 + (C-6B)x + (4A-6C)$$

Example 3: Write $\frac{f}{g} = \frac{x^2+3x-4}{x^3-6x^2+4x-24}$ as a sum of partial fractions.

Hint: $g(6) = 0$.

$x^3-6x^2+4x-24 = (x-6)(x^2+4)$, so we are looking for

$$\frac{x^2+3x-4}{(x-6)(x^2+4)} = \frac{A}{x-6} + \frac{Bx+C}{x^2+4} = \boxed{\frac{5/4}{x-6} + \frac{\frac{-1}{4}x+\frac{3}{2}}{x^2+4}}$$

Answer

$$1x^2+3x-4 = (A+B)x^2 + (C-6B)x + (4A-6C)$$

$$\begin{cases} A+B = 1 \\ C-6B = 3 \\ 4A-6C = -4 \end{cases} \longrightarrow \begin{aligned} A &= 5/4 \\ B &= -1/4 \\ C &= 3/2 \end{aligned}$$

Your turn!

Write $\dfrac{f(x)}{g(x)} = \dfrac{x + 14}{x^2 - 2x - 8}$ as a sum of partial fractions.

Try it yourself at

https://itempool.com/theadamabrams/c/l7s7UUu8WrU

Answer at itempool.com/theadamabrams/live

GO LIVE

☐ Collect student names

If checked, students will be required to enter their names, which will then be associated with their responses.

Show advanced settings

Which topic(s) do you find **most confusing**? Do not select more than 3.

- ☐ high school fractions, square roots, algebra, etc.
- ☐ cos, sin, radians (like π/6)
- ☐ complex numbers in rectangular form
- ☐ complex numbers in polar form
- ☐ complex numbers on a graph/picture
- ☐ irreducible polynomials
- ☐ multiplicity of a zero
- ☐ polynomial quotient and remainder
- ☐ partial fractions
- ☐ nothing (because everything is easy)

1 Unstarted — 2 Accepting answers — 3 Results

Which topic(s) do you think you **understand well**?

- ☐ complex numbers in rectangular form
- ☐ complex numbers in polar form
- ☐ complex numbers on a graph/picture
- ☐ irreducible polynomials
- ☐ multiplicity of a zero
- ☐ polynomial quotient and remainder
- ☐ partial fractions

1 Unstarted — 2 Accepting answers — 3 Results

Fun (?) activity

Activity 0:

Step 1. Pick a complex number z.

Step 2. $z_{\text{new}} = (z_{\text{old}})^2$.

Step 3. Repeat Step 2 forever.

Activity -1:

Step 1. Pick a complex number z.

Step 2. $z_{\text{new}} = (z_{\text{old}})^2 - 1$.

Step 3. Repeat Step 2 forever.

What happens? Does your list of new z-values get very big? Very close to zero? Neither? It depends on your starting number.

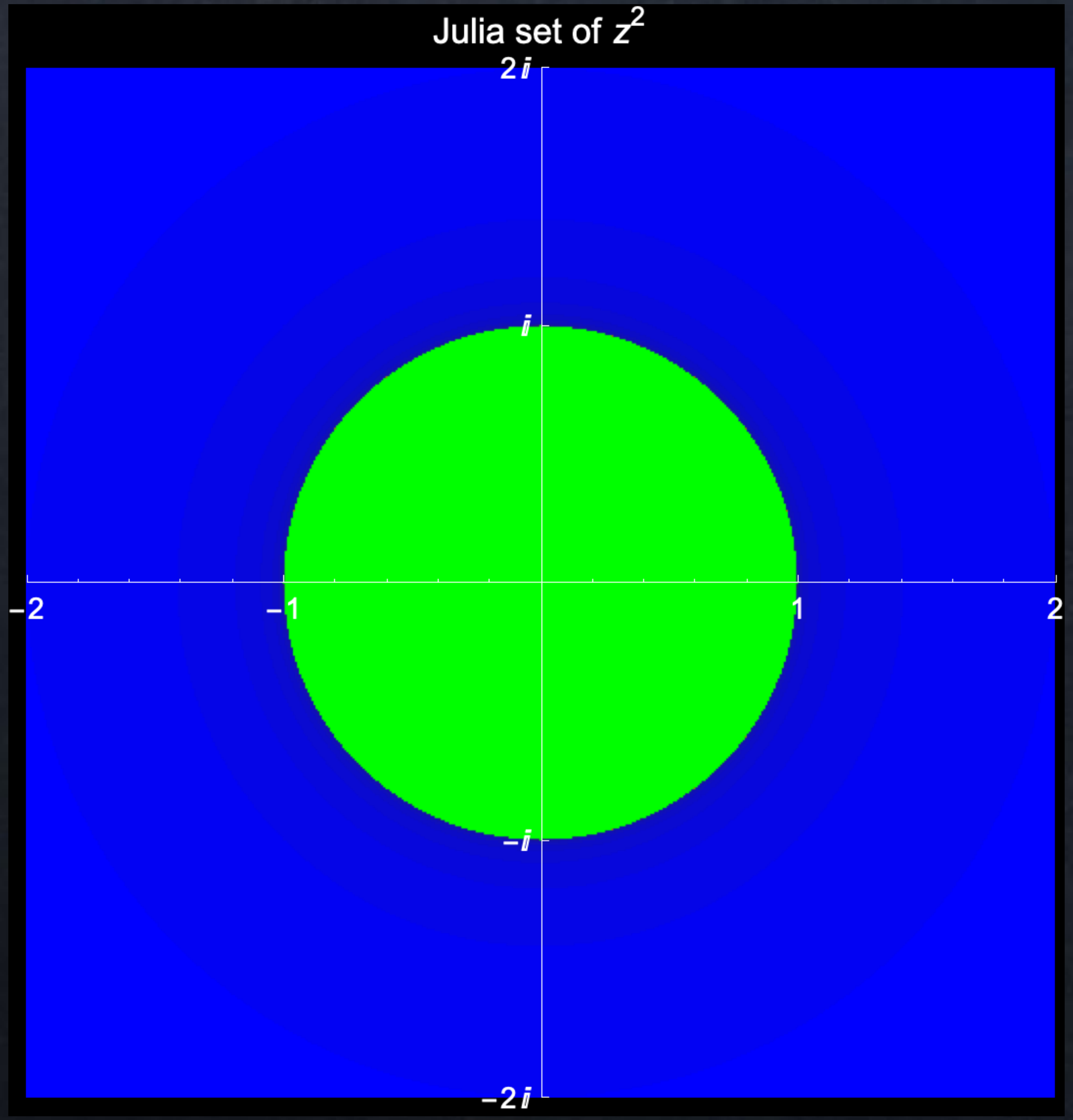
Fun (?) activity

Activity 0:

Step 1. Pick a complex number z.

Step 2. $z_{\text{new}} = (z_{\text{old}})^2$.

Step 3. Repeat Step 2 forever.



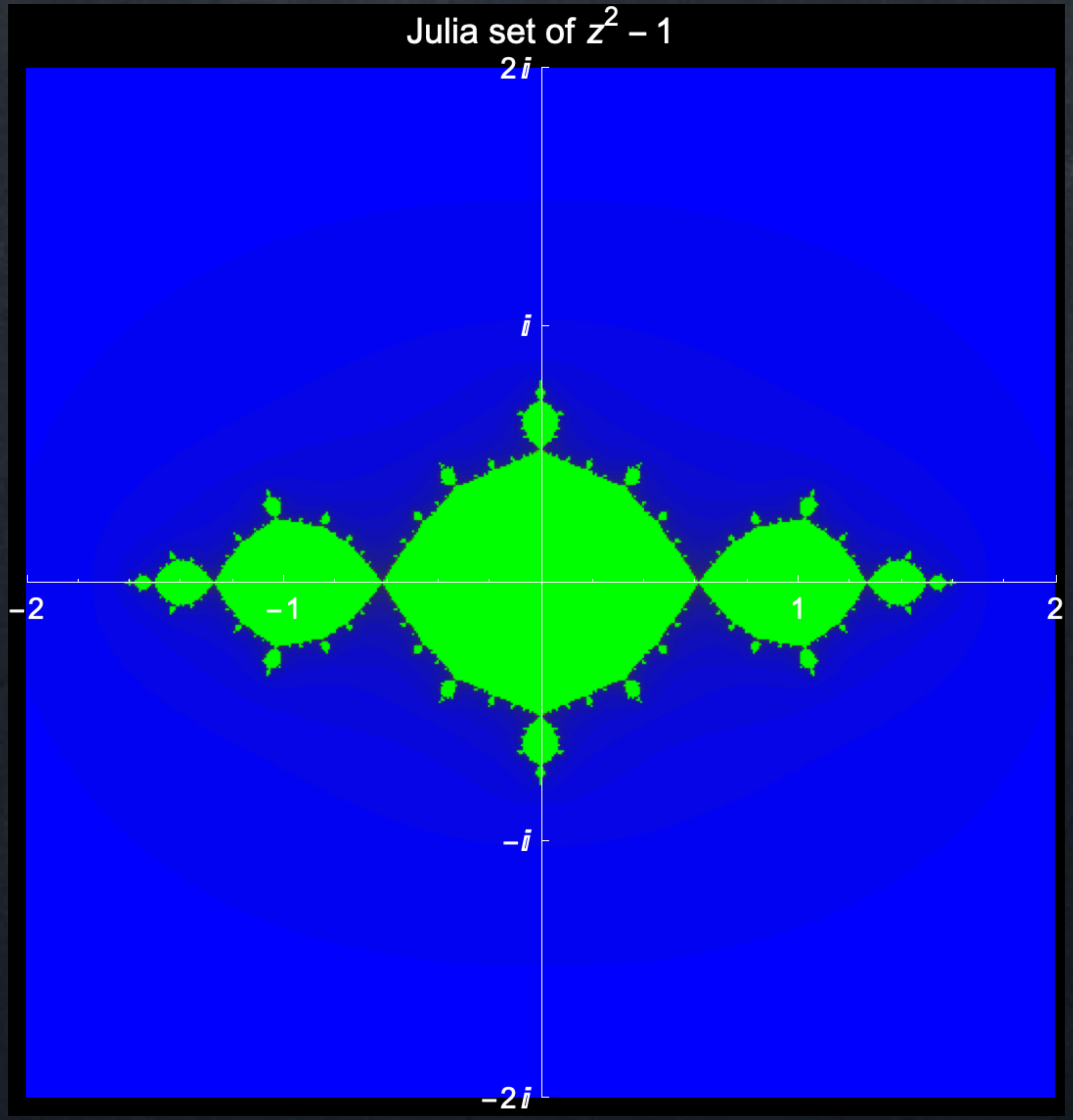
Activity -1:

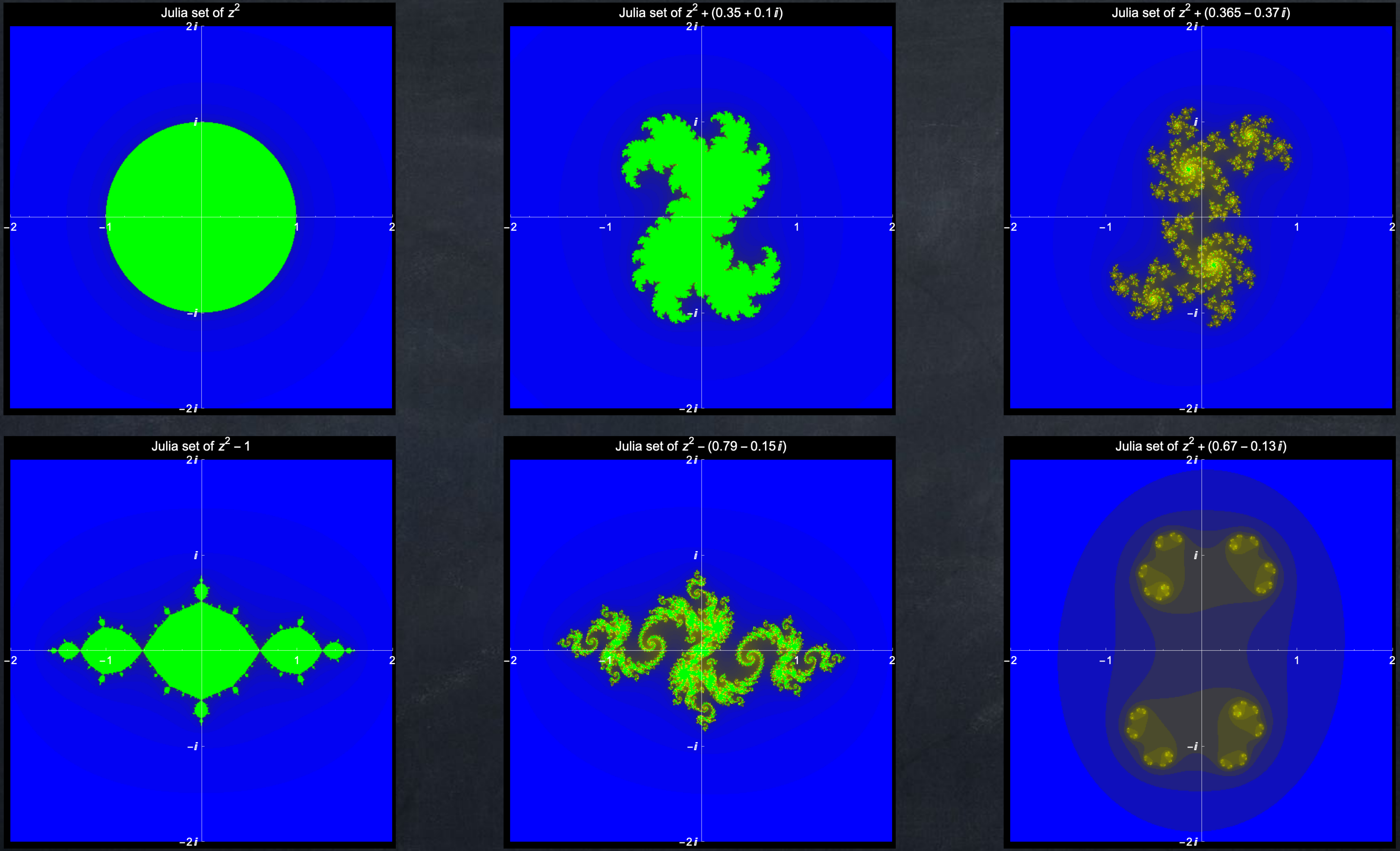
Step 1. Pick a complex number z.

Step 2. $z_{\text{new}} = (z_{\text{old}})^2 - 1$.

Step 3. Repeat Step 2 forever.

Green points are where z is close to 0 after many loops.





This **not** part of Math 1688. It's just an example of how there is a lot more to complex numbers and polynomials than we can cover in this class.