Math 1688

18 November 2021

Warm-up:
What have you learned about vectors from other classes?

theadamabrams.com/live

This class

- Complex numbers
  - Rectangular form
  - Polar form
- Polynomials
  - Factoring
    - Irreducible polyn.
    - Roots (zeros)
      - Multiplicities
  - Quotient and remainder
- Vectors
- Matrices

not linear algebra

linear algebra

Vectors as Lists / Points / Arrows

The word “vector” can mean many things.
At times we will think of a **vector** as

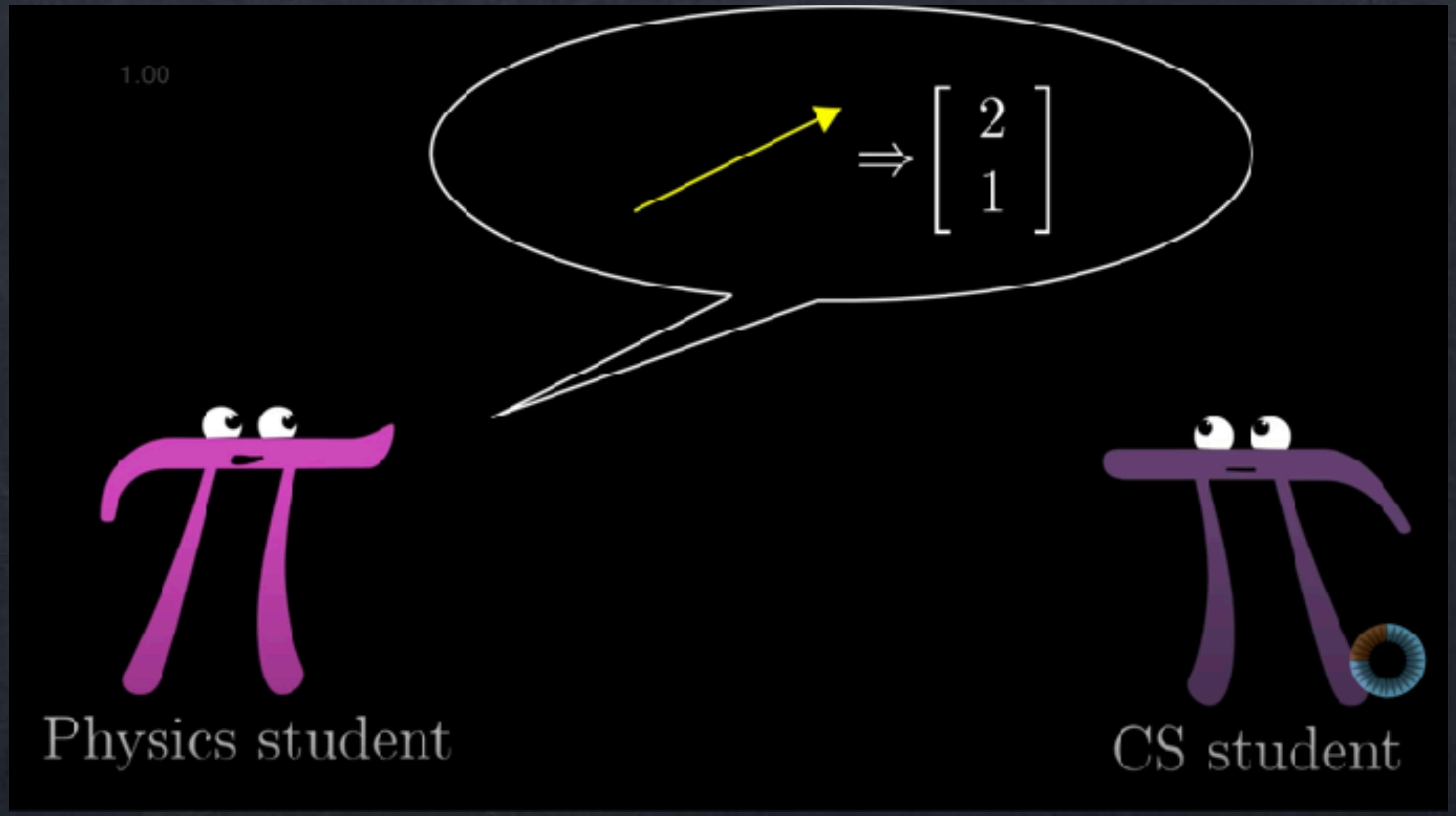
- a list.
- a point.
- an arrow with its tail at the origin.
- an arrow with its tail anywhere.

There is another option:

- an element of an abstract vector space,

but we won’t use that idea of a vector in this class.

youtu.be/fNk_zzaMoSs (3B1B)



Vectors as Lists
Points
Arrows

A **vector** is a list of numbers.

- We can write the same list of numbers in many formats. For example,

$$(5, 3, 8) \qquad \langle 5, 3, 8 \rangle \qquad [\,5 \;\; 3 \;\; 8\,] \qquad \begin{bmatrix} 5 \\ 3 \\ 8 \end{bmatrix} \qquad \begin{pmatrix} 5 \\ 3 \\ 8 \end{pmatrix}$$

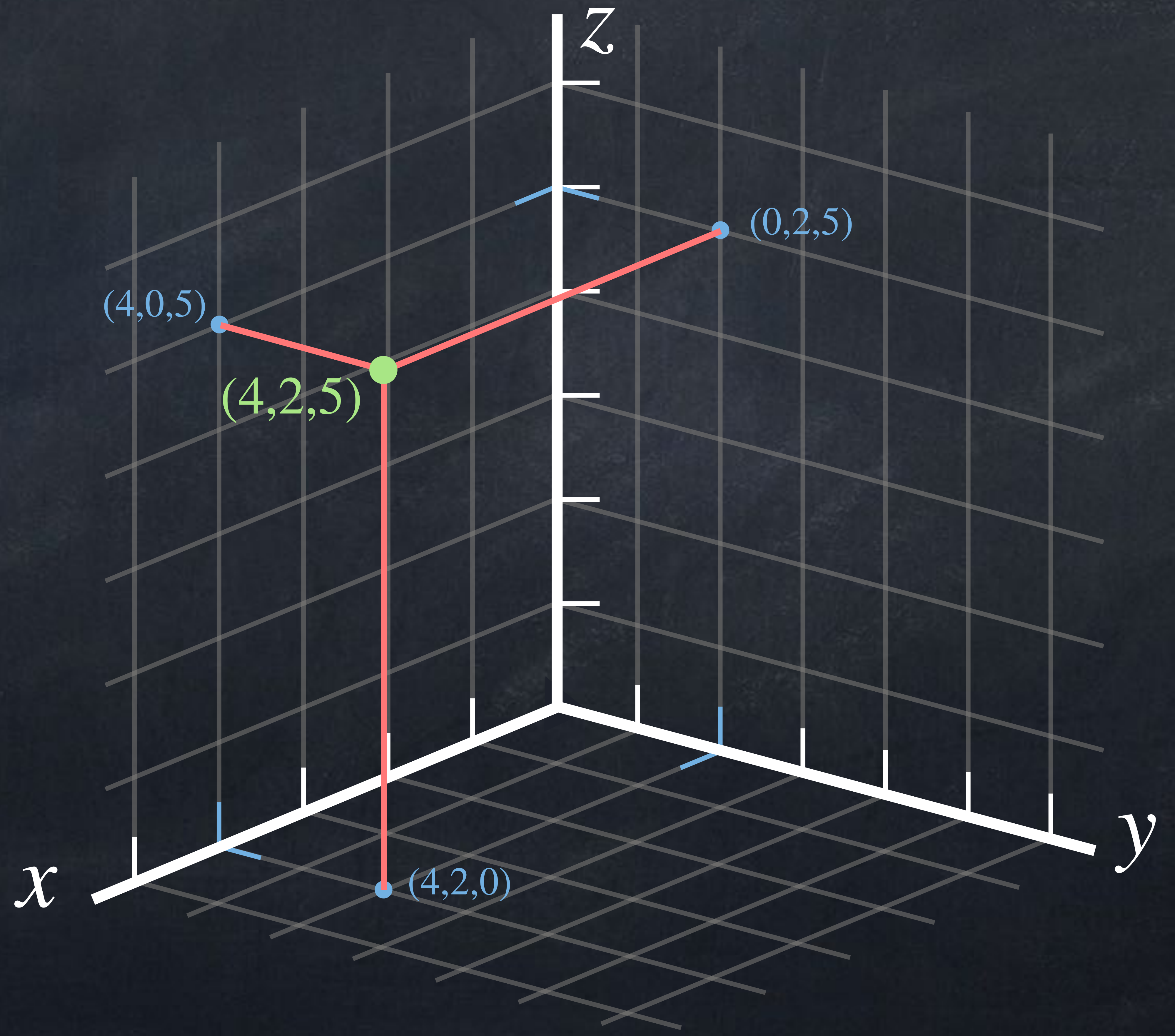
  are all exactly the same vector.
- Each numbers is a **component** of the vector.
  - For $[5,3,8]$, the "1st component" is 5, the "2nd component" is 3, etc.
- We often label the components with subscripts: $\overrightarrow{u} = \langle u_1, u_2, u_3 \rangle$. But sometimes we instead label a whole vector this way: $\overrightarrow{u_1}$ and $\overrightarrow{u_2}$.

Vectors as Points

Lists

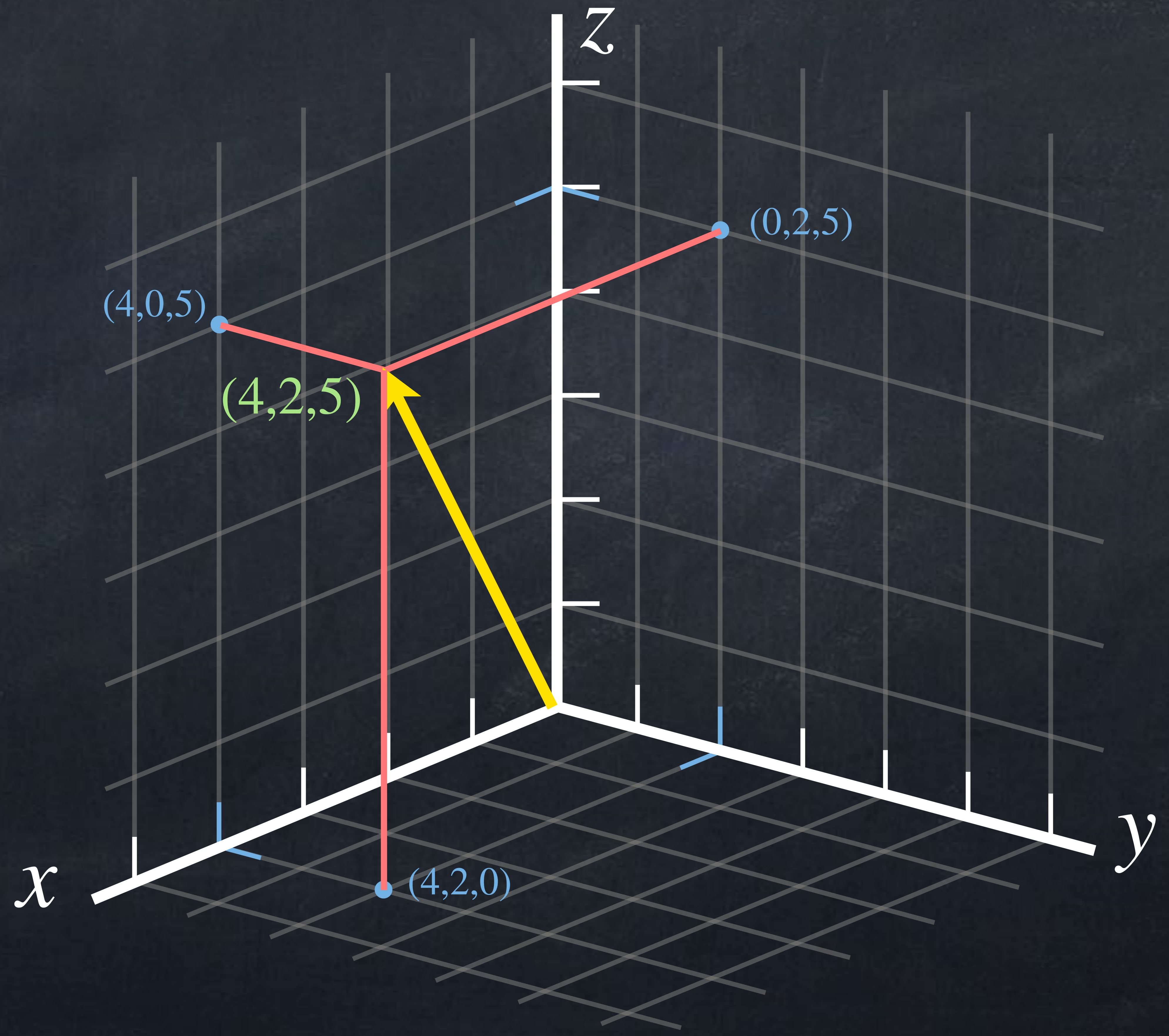
Arrows

A **vector** is a point (a dot) in 2D or 3D space.



Vectors as Lists Points Arrows

A **vector** is something that has a magnitude and a direction.
In other words, it is an arrow.



Lists
Points

Vectors as Arrows

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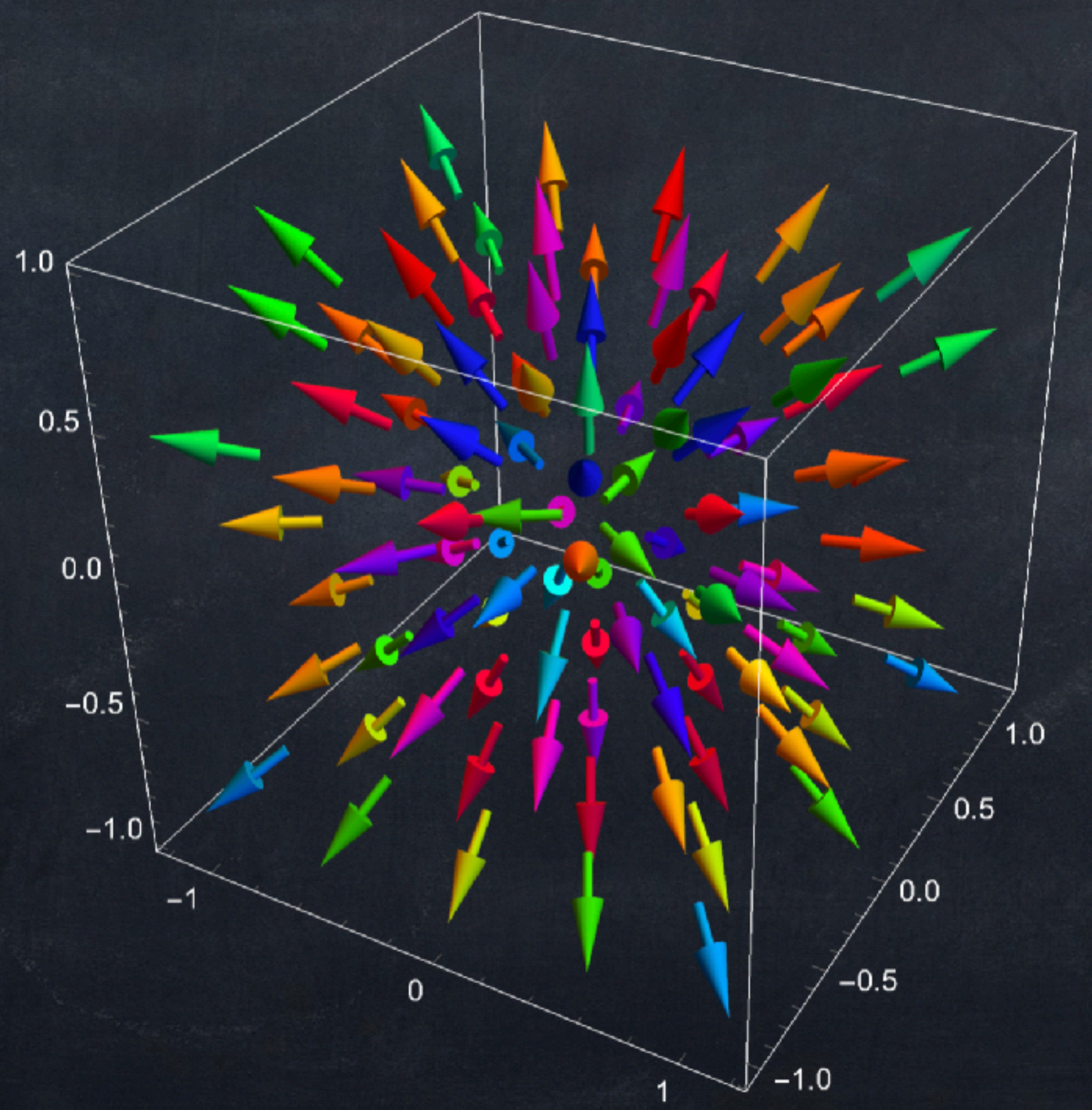


Vector variables

In different text/videos, a vector variable might be written as any of these:

$$\overrightarrow{u} \qquad \overrightarrow{u} \qquad \boldsymbol{u} \qquad \mathbf{u} \qquad u$$

Often we use letters u, v, w or a, b, c for vectors.

If the vector has a specific meaning, we might use a letter related to that meaning (for example, $\overrightarrow{n}$ for a "normal vector").

Vector variables

Two vectors are **equal** if their first components are equal and their second components are equal and so on.

- Example: $\langle 5, 1, 9 \rangle = \langle 2+3, \frac{6}{6}, 13-4 \rangle$

As with numbers, sometimes an equation describes one specific value

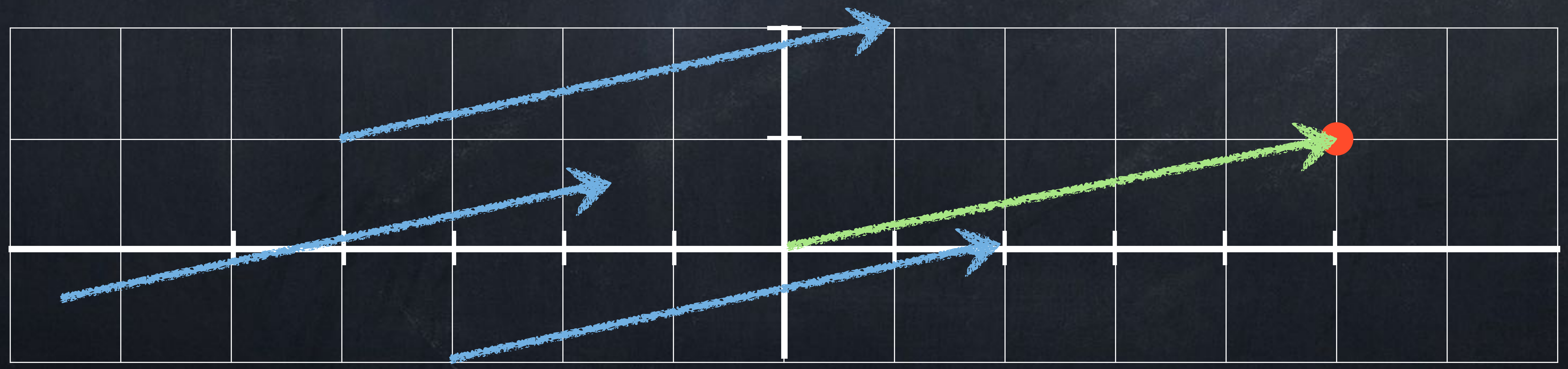
- $\vec{u} = \langle 1, -3 \rangle$
- $x = -8$
- and sometimes there are many values that make an equation true:
- $|\vec{u}| = 5$
- $x^2 - 4x + 3 = 0$

Origin

The **zero vector** is $\overrightarrow{0} = \langle 0,0 \rangle$ in 2D and $\overrightarrow{0} = \langle 0,0,0 \rangle$ in 3D.

Depending on context, a vector like $\langle 5,1 \rangle$ might refer to

- *any* arrow that points in a direction 5 right and 1 up, or
- the specific arrow from $(0,0)$ to $(5,1)$, or
- the point $(5,1)$.

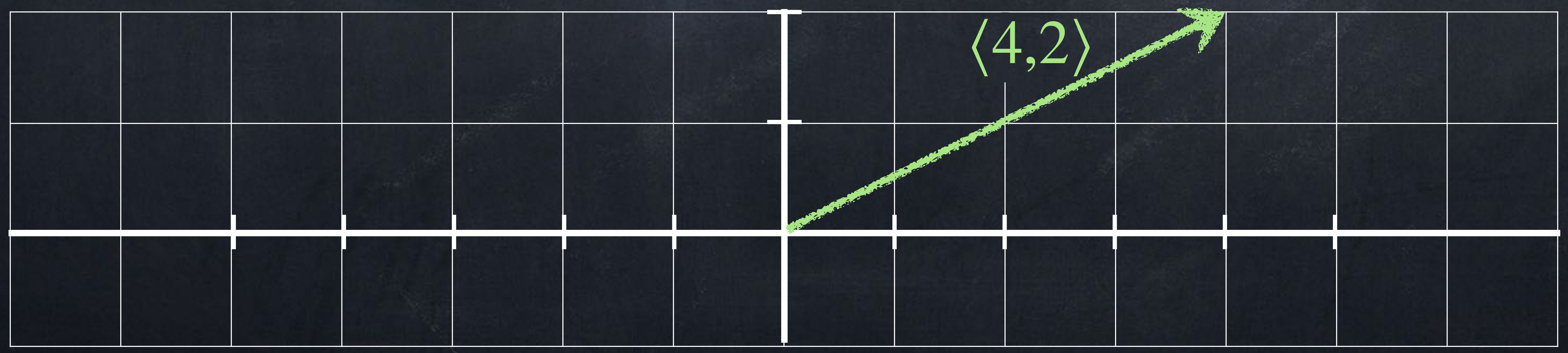


Magnitude

The **magnitude** (or **length** or **norm**) of the vector $\overrightarrow{v} = \langle v_1, v_2, \ldots, v_n \rangle$ is

$$|\overrightarrow{v}| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

In 2D or 3D, this is exactly the physical length of the arrow, or the length of the line segment from the origin to the point $\overrightarrow{v}$.



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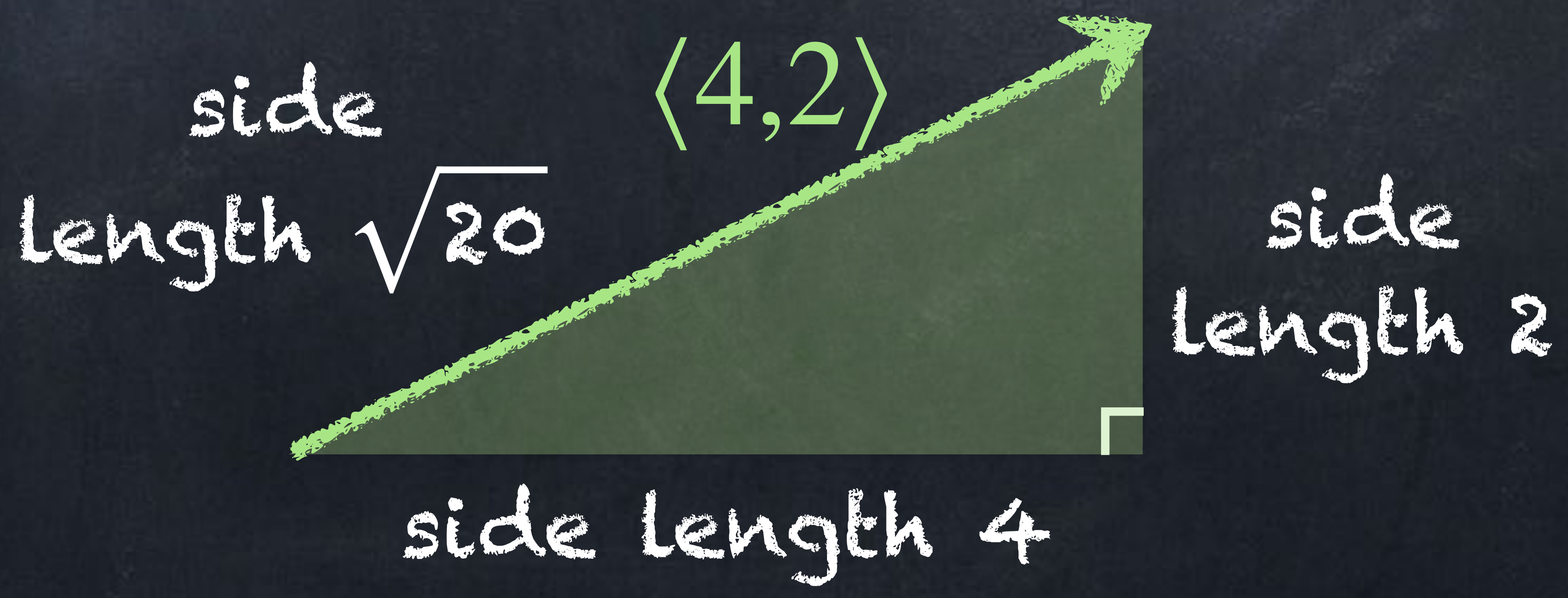


Scalar multiplication

For this class, a **scalar** is a number (that is, not a vector).

Given a scalar s and a vector $\overrightarrow{v} = \langle v_1, v_2, \ldots, v_n \rangle$, we can multiply s and $\overrightarrow{v}$ to get

$$s\overrightarrow{v} = \langle sv_1, sv_2, \ldots, sv_n \rangle.$$

Examples:

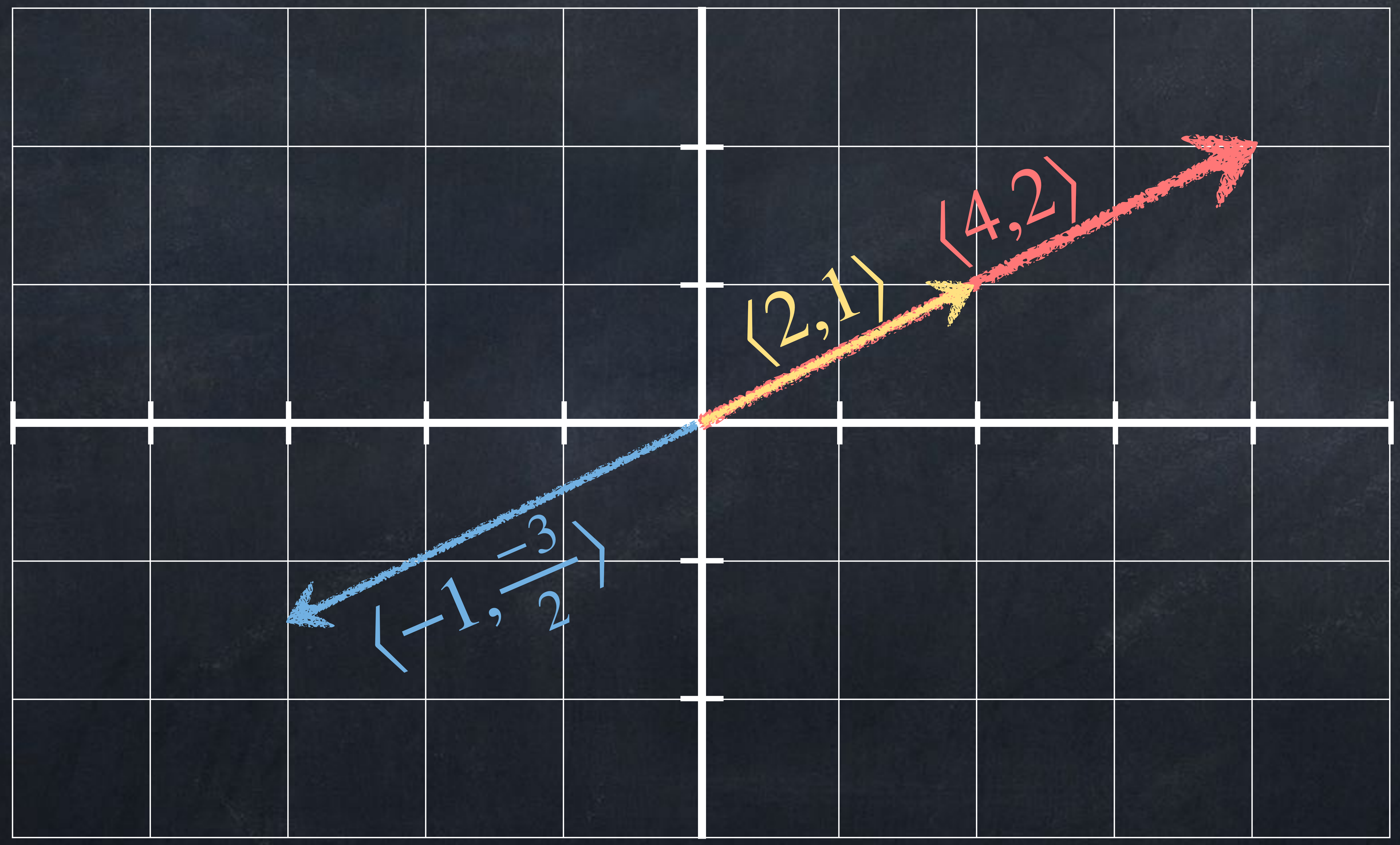
- $3\langle 8,1 \rangle = \langle 24,3 \rangle$
- $\frac{1}{2}\langle 8,1 \rangle = \langle 4,\frac{1}{2} \rangle$
- $4\langle -3, 9.1 \rangle = \langle -12, 36.8 \rangle$
- $-2\langle 5, -4 \rangle = \langle -10,8 \rangle$
- $0\langle 5,7 \rangle = \langle 0,0 \rangle$

We say that $\langle 24,3 \rangle$ is a **scalar multiple** of $\langle 8,1 \rangle$.

Note that $\langle 24,10 \rangle$ is *not* a scalar multiple of $\langle 8,1 \rangle$.

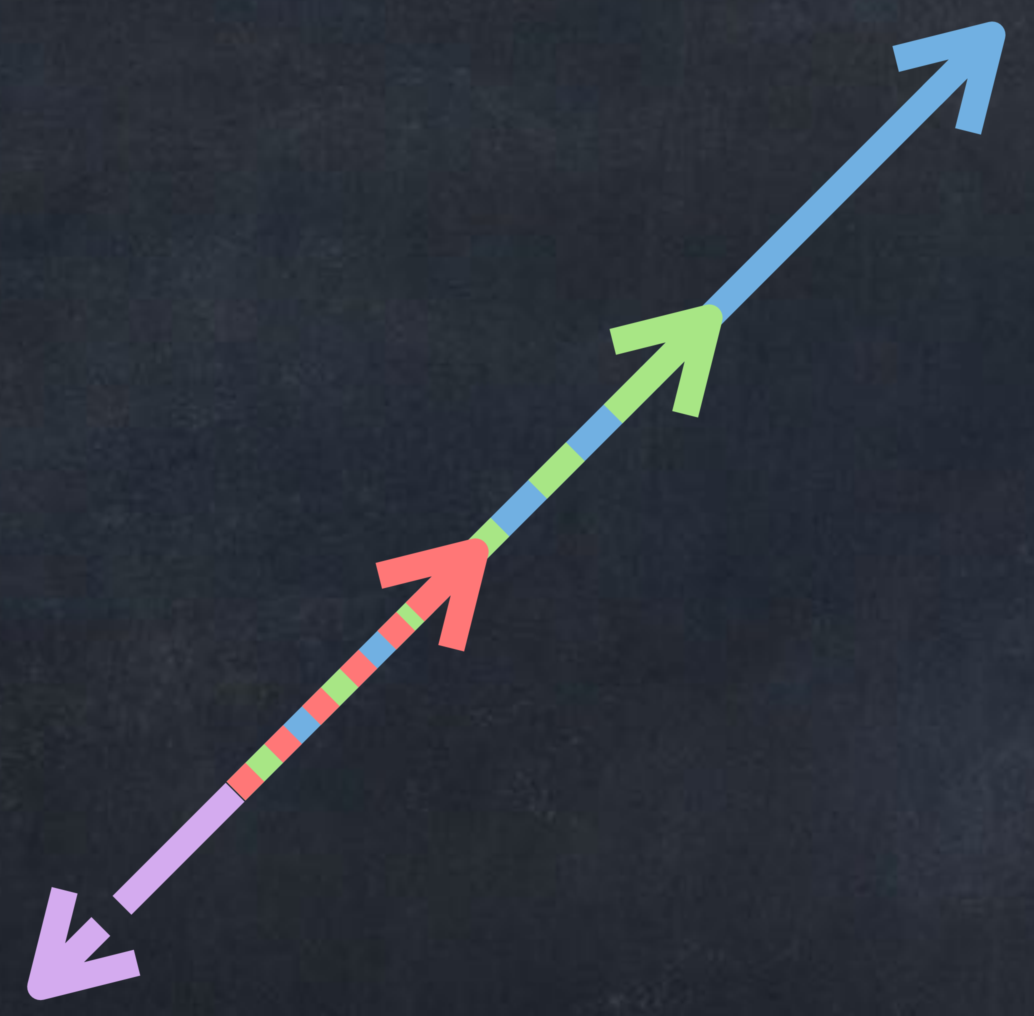
Scalar multiplication

Geometrically, $s\overrightarrow{v}$ is a “stretched” version of $\overrightarrow{v}$.



Parallel vectors

Two vectors $\overrightarrow{u}$ and $\overrightarrow{v}$ are **parallel** if $\overrightarrow{u} = s\overrightarrow{v}$ for some $s \neq 0$.



- Some people require $s > 0$ and say, for example, that $\langle 3,2 \rangle$ and $\langle -6, -4 \rangle$ are “anti-parallel”. Some people call them parallel.
- For any $\overrightarrow{v}$, $\overrightarrow{0}$ *is* a scalar multiple of $\overrightarrow{v}$, but $\overrightarrow{0}$ is not parallel to $\overrightarrow{v}$.

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Vector addition

As lists, vectors are added by adding each coordinate.

- Example: $\langle 9, -4 \rangle + \langle 5, 6 \rangle = \langle 14, 2 \rangle$
- Example: $$\begin{bmatrix} 5 \\ 8 \end{bmatrix} + \begin{bmatrix} -2 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \\ 15 \end{bmatrix}$$

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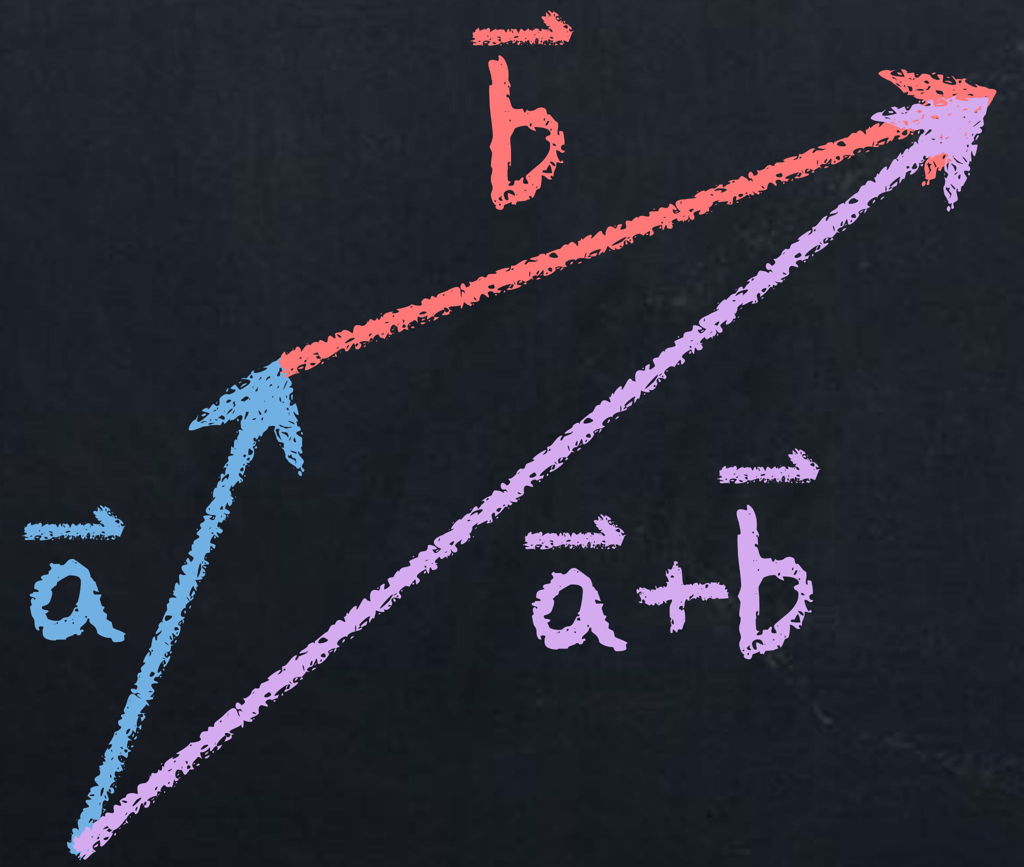

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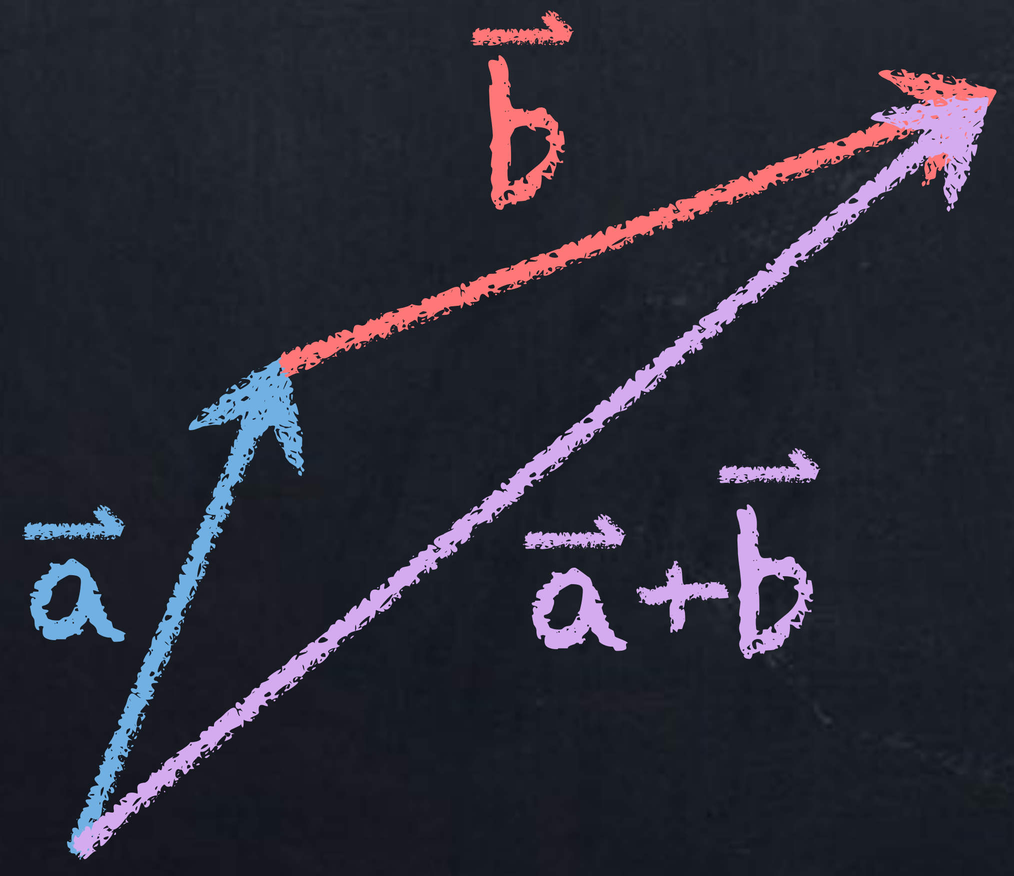

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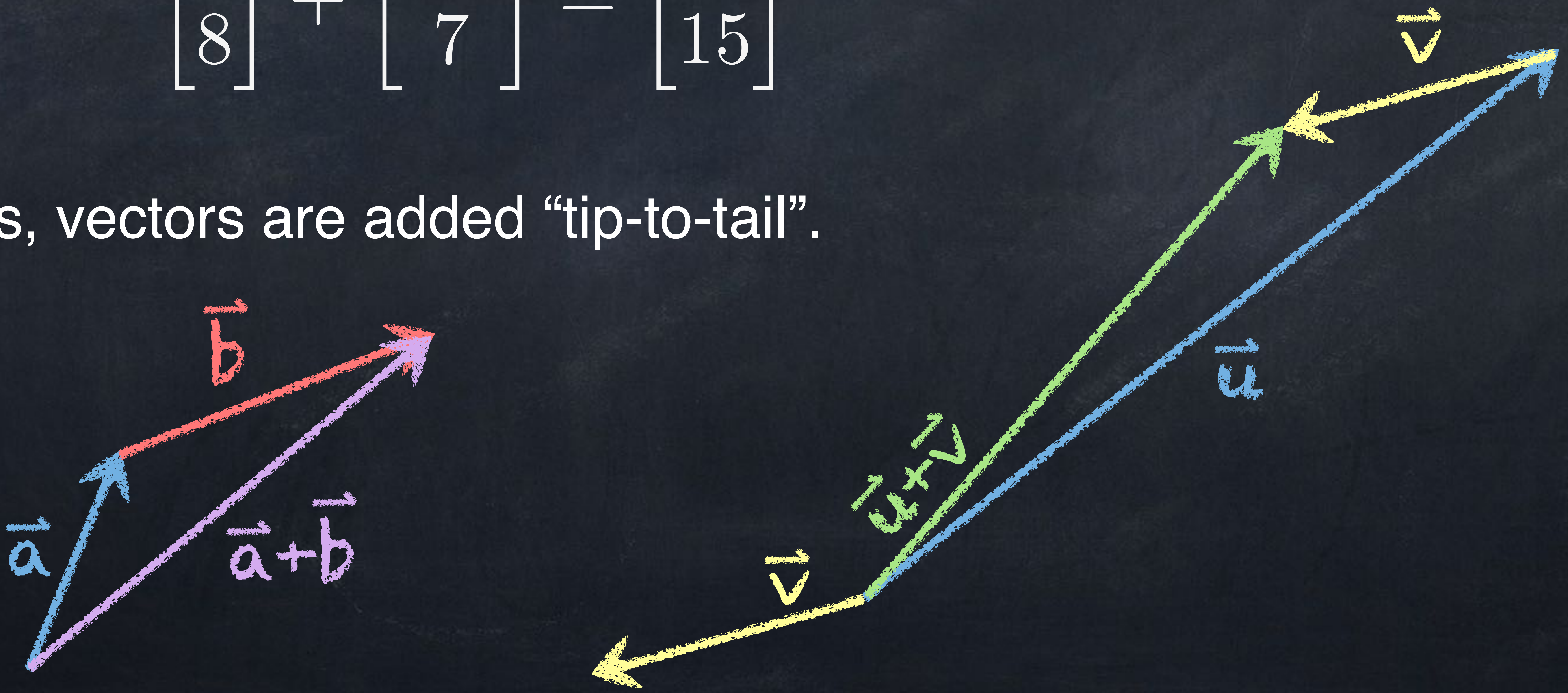

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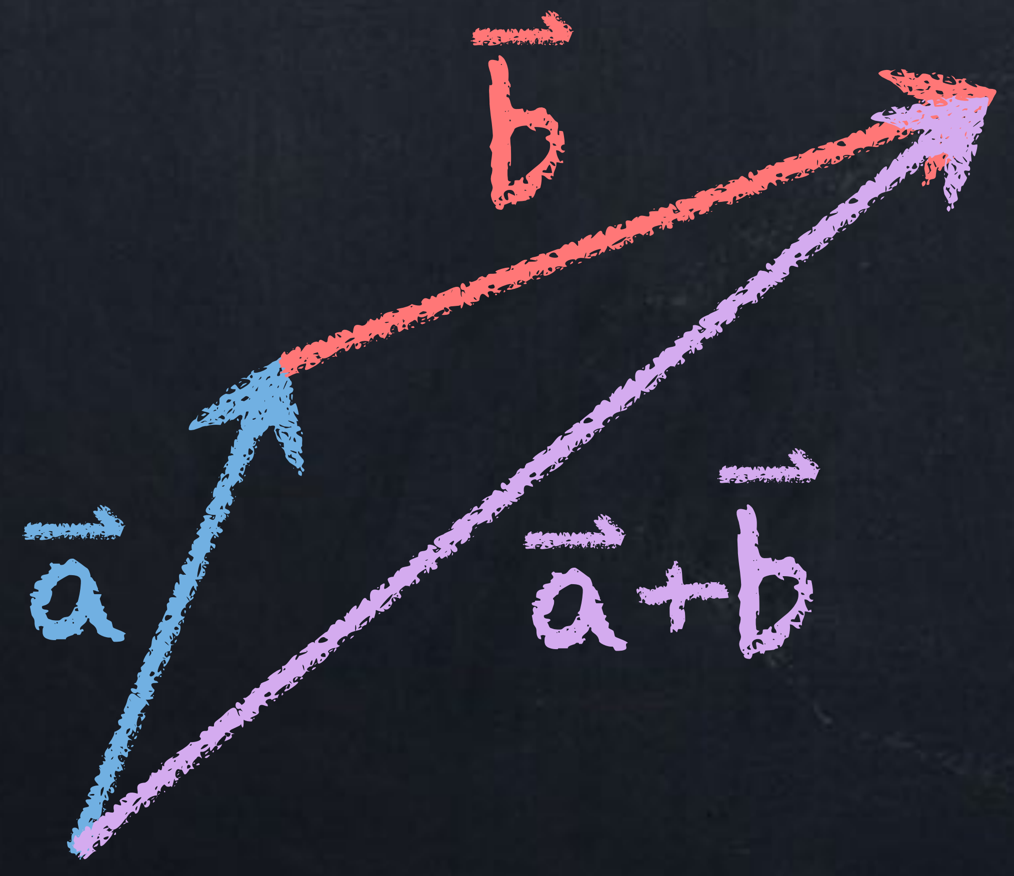

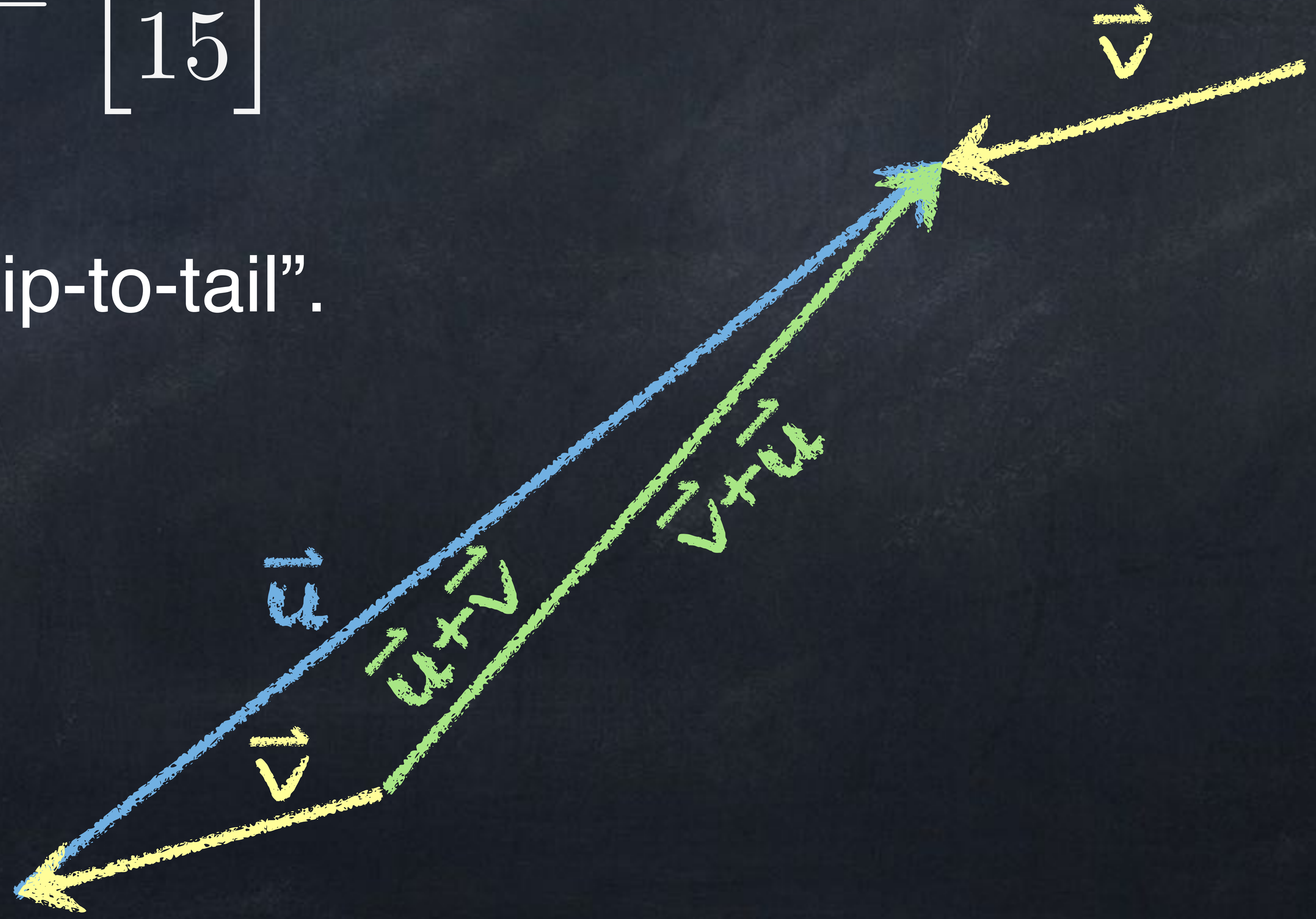

Vector identities

$$\overrightarrow{u} + \overrightarrow{v} = \overrightarrow{v} + \overrightarrow{u}$$

$$(\overrightarrow{u} + \overrightarrow{v}) + \overrightarrow{w} = \overrightarrow{u} + (\overrightarrow{v} + \overrightarrow{w})$$

$$s(t\overrightarrow{v}) = (st)\overrightarrow{v}$$

$$s(\overrightarrow{u} + \overrightarrow{v}) = (s\overrightarrow{u}) + (s\overrightarrow{v})$$

$$(s + t)\overrightarrow{v} = (s\overrightarrow{v}) + (t\overrightarrow{v})$$

$$\overrightarrow{u} + \overrightarrow{0} = \overrightarrow{u}$$

$$\overrightarrow{u} + (-\overrightarrow{u}) = \overrightarrow{0}$$

Vector identities

$$\overrightarrow{a} + \overrightarrow{b} = \overrightarrow{b} + \overrightarrow{a}$$

$$(\overrightarrow{a} + \overrightarrow{b}) + \overrightarrow{c} = \overrightarrow{a} + (\overrightarrow{b} + \overrightarrow{c})$$

$$k(m\overrightarrow{a}) = (km)\overrightarrow{a}$$

$\ldots$

Basis vectors

Later, we will talk about the general idea of a "basis", but for now we will use just one 2D example and one 3D example.

- In 2D, the **standard basis vectors** are

$$\vec{\imath} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \vec{\jmath} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

- In 3D, the **standard basis vectors** are

$$\vec{\imath} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \vec{\jmath} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } \vec{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

We can write *any* vector using scalar multiples, these basis vectors, and vector addition.

Basis vectors

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Examples:

- $$\begin{bmatrix}5\\2\end{bmatrix} = \begin{bmatrix}5\\0\end{bmatrix} + \begin{bmatrix}0\\2\end{bmatrix} = 5\begin{bmatrix}1\\0\end{bmatrix} + 2\begin{bmatrix}0\\1\end{bmatrix} = 5\vec{\imath} + 2\vec{\jmath}$$
- $$\begin{bmatrix}6\\0.91\\-2\end{bmatrix} = 6\vec{\imath} + 0.91\vec{\jmath} - 2\overrightarrow{k}$$
- $$\begin{bmatrix}a\\b\end{bmatrix} = a\vec{\imath} + b\vec{\jmath}$$
- $$\begin{bmatrix}4\\0\\1\end{bmatrix} = 4\vec{\imath} + \overrightarrow{k}$$
- $$\begin{bmatrix}5\\2\\0\end{bmatrix} = 5\vec{\imath} + 2\vec{\jmath}$$

Vector subtraction

using coordinates

geometrically

Vector subtraction

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- Example: $\langle 9, -4\rangle - \langle 5, 6\rangle = \langle 4, -10\rangle$
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What does $\overrightarrow{u} - \overrightarrow{v}$ mean geometrically?

- We *could* first find the scalar multiple $-\overrightarrow{v} = (-1)\overrightarrow{v}$ and then use tip-to-tail addition to find $\overrightarrow{u} + (-\overrightarrow{v})$.
- What does $a - b$ mean for numbers?

New meanings for old things

What does 5×3 mean?

- 
- More advanced: no pictures, just $5 + 5 + 5$.

What does $5 \times \frac{1}{3}$ mean? 5×9.2 ? $7.65 \times (-12)$?

- **Multiplication can have different meanings or interpretations.**
  - This is also true for subtraction.

Subtraction

What does $5 - 3$ mean **on a number line**?

Answer: The number $5 - 3$ describes how to move from 3 to 5.

In general, $b - a$ describes how to move from a to b.

Subtraction

What does $5 - 3$ mean **on a number line**?

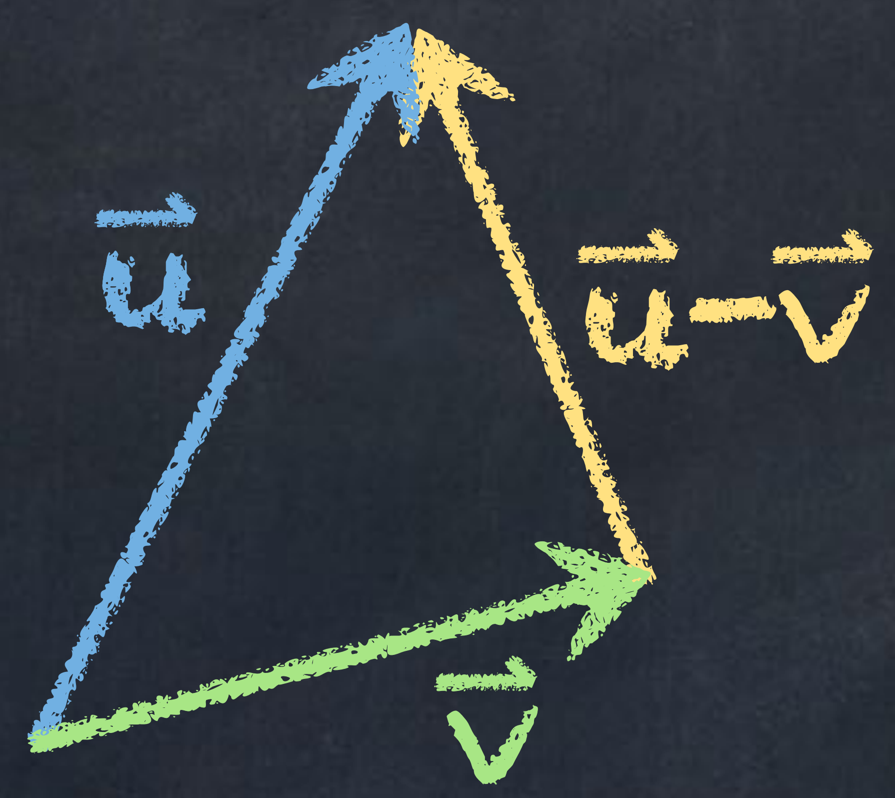
Answer: The number $5 - 3$ describes how to move from 3 to 5.

- To go from 5 to 3 instead, we move *left*, which is why $3 - 5$ is negative.

In general, $b - a$ describes how to move from a to b.

Vector subtraction

The vector $\overrightarrow{u} - \overrightarrow{v}$ points from the tip of $\overrightarrow{v}$ to the tip of $\overrightarrow{u}$.



Note: The tails (start) of u⃗ and v⃗ must be at the same place to use this method.

- This agrees with finding $\overrightarrow{u} - \overrightarrow{v}$ by adding $\overrightarrow{u} + (-\overrightarrow{v})$ tip-to-tail.

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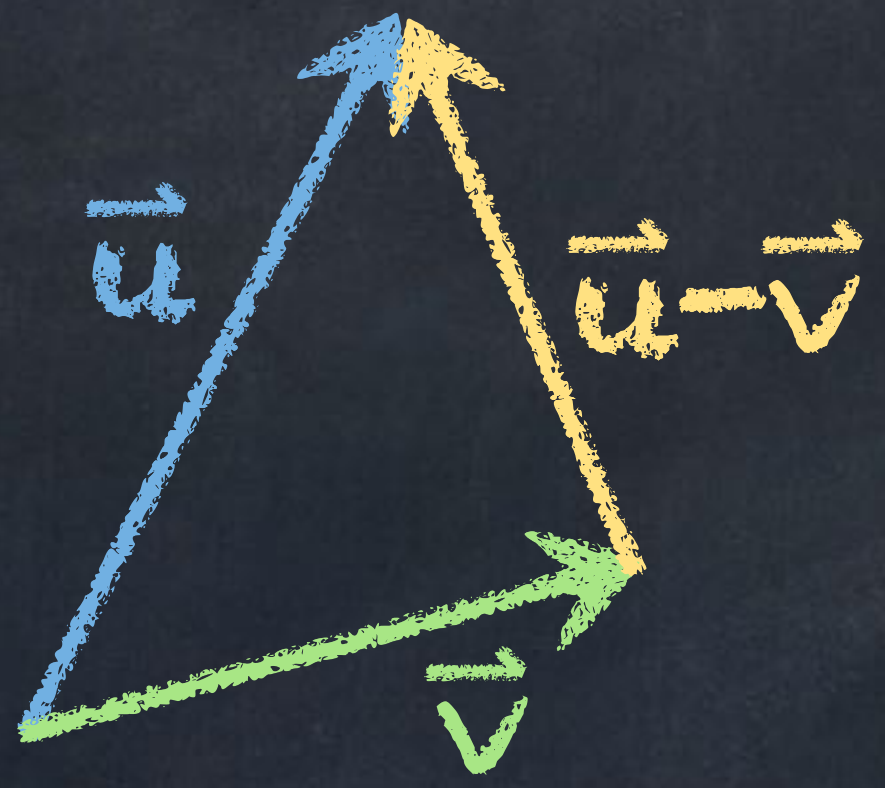

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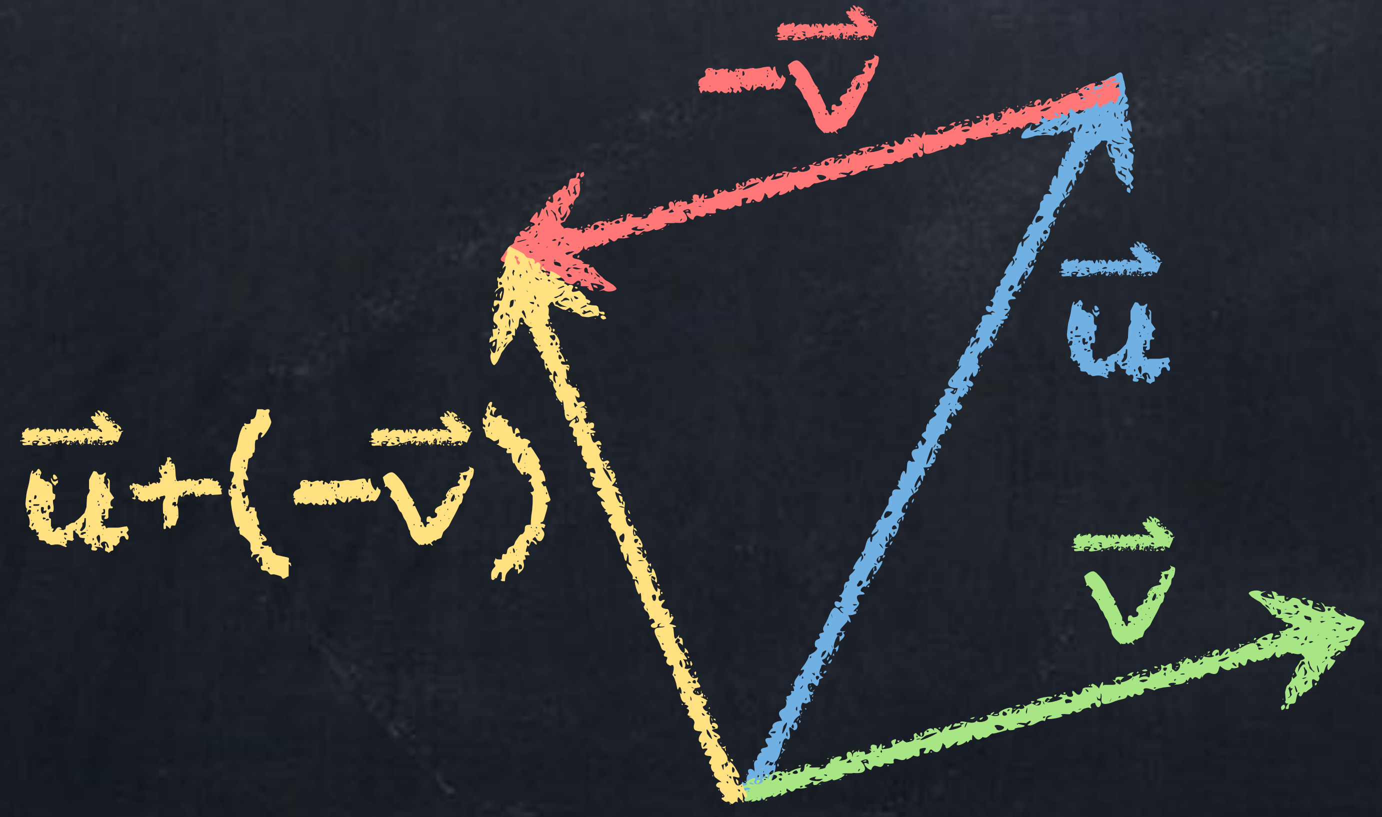

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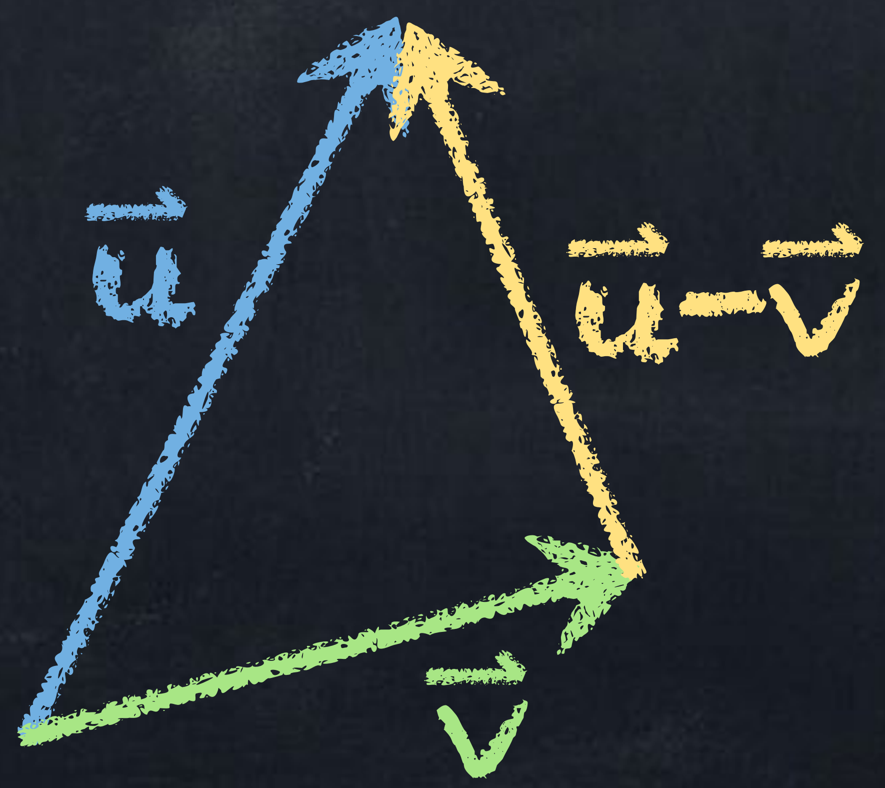

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Multiplication of vectors

Combining $\overrightarrow{a} = \langle a_1, a_2 \rangle$ and $\overrightarrow{b} = \langle b_1, b_2 \rangle$ into $\langle a_1 b_1, a_2 b_2 \rangle$ is not actually useful. We will never do this.

Instead, we have "dot product" and "cross product" of vectors.

- $\overrightarrow{a} \cdot \overrightarrow{b}$ can be done for vectors of any dimension.
- $\overrightarrow{a} \times \overrightarrow{b}$ will only be done in 3D.
- We will never write $\overrightarrow{a}\,\overrightarrow{b}$ without either $\cdot$ or $\times$.

Dot product

The **dot product** (or **inner product** or **scalar product**) of $\overrightarrow{a} = \langle a_1, a_2, \ldots, a_n \rangle$ and $\overrightarrow{b} = \langle b_1, \ldots, b_n \rangle$ is

$$\overrightarrow{a} \cdot \overrightarrow{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

This is a number, not a vector.

Examples:

- $\langle 5,7 \rangle \cdot \langle 8,2 \rangle = 40 + 14 = 54$
- $\langle 3, -1,8 \rangle \cdot \langle 0,4,2 \rangle = 0 + -8 + 16 = 8$
- $\langle x, 6 \rangle \cdot \langle 5,2 \rangle = 5x + 12$
- $\langle 2, \ -1 \rangle \cdot \langle t - 9,t \rangle = 2(t - 9) - t = t - 18$

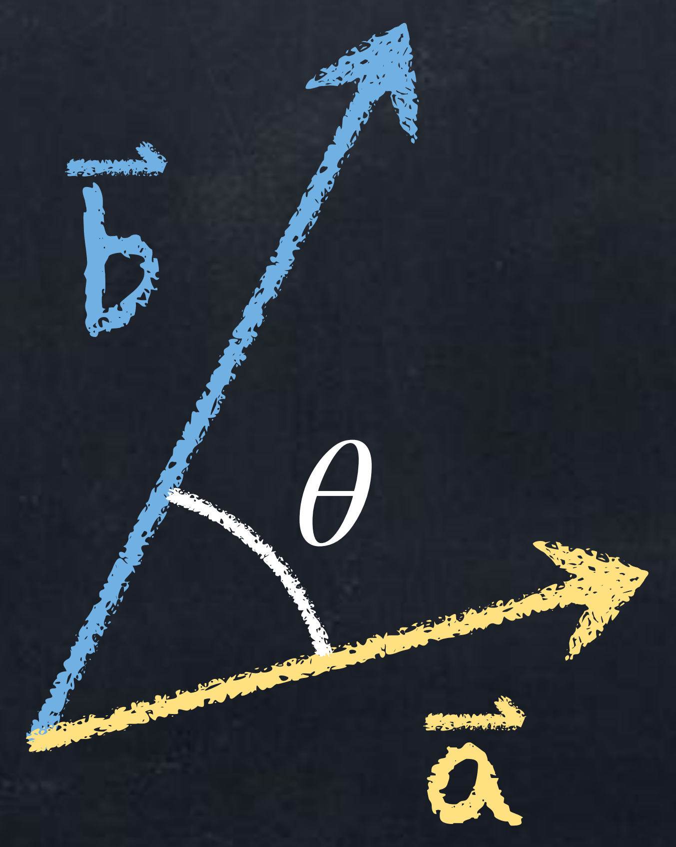
Dot product

The dot product has another formula:

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

where θ is the angle between $\vec{a}$ and $\vec{b}$.

Using both of these formulas, we can find the angle between vectors.



Example: Find the acute angle between $\langle\sqrt{3},1\rangle$ and $\langle 0,7\rangle$.

$\vec{a}$ $\vec{b}$

$$|\vec{a}| = \sqrt{(\sqrt{3})^2 + (1)^2} = \sqrt{3+1} = \sqrt{4} = 2$$

$$|\vec{b}| = \sqrt{(7)^2 + (0)^2} = \sqrt{49} = 7$$

$$\vec{a}\cdot\vec{b} = (\sqrt{3})(0) + (1)(7) = 7$$

Since $\vec{a}\cdot\vec{b}$ is also $(2)(7)\cos\theta$, we know

$$(2)(7)\cos\theta = 7 \quad\rightarrow\quad \cos\theta = 1/2 \quad\rightarrow\quad \theta = 60^\circ$$

Example: Find the acute angle between $\langle 4,3 \rangle$ and $\langle 1,6 \rangle$.

$$|\vec{a}| = \sqrt{16+9} = \sqrt{25} = 5$$

$$|\vec{b}| = \sqrt{1+36} = \sqrt{37}$$

$$\vec{a} \cdot \vec{b} = (4)(1) + (2)(6) = 4 + 12 = 16$$

So $5\sqrt{37}\cos\theta = 16$

$\cos\theta = \dfrac{16}{5\sqrt{37}}$ cannot be solved by hand.

We say $\theta = \arccos\left(\dfrac{16}{5\sqrt{37}}\right)$ or $\cos^{-1}\left(\dfrac{16}{5\sqrt{37}}\right)$.