Math 1688

25 November 2021

Warm-up:
Vector subtraction, perpendicular vectors, parallel vectors.

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Last week

The word “vector” can mean many things.
At times we will think of a vector as

- a list.
- a point.
- an arrow with its tail at the origin.
- an arrow with its tail anywhere.

There is another option,

- an element of an abstract vector space,

but we won’t use that idea of a vector in this class.

Last week

Vector variables can be written with an arrow above the letter.

$\overrightarrow{u}$ $\vec{u}$

If we write a vector as a list of numbers, there are many possible formats:

$(5, 3, 8)$

$[\,5 \;\; 3 \;\; 8\,]$

$\langle 5, 3, 8 \rangle$

$\begin{bmatrix} 5 \\ 3 \\ 8 \end{bmatrix}$ $\begin{pmatrix} 5 \\ 3 \\ 8 \end{pmatrix}$

Special vectors in 2D:

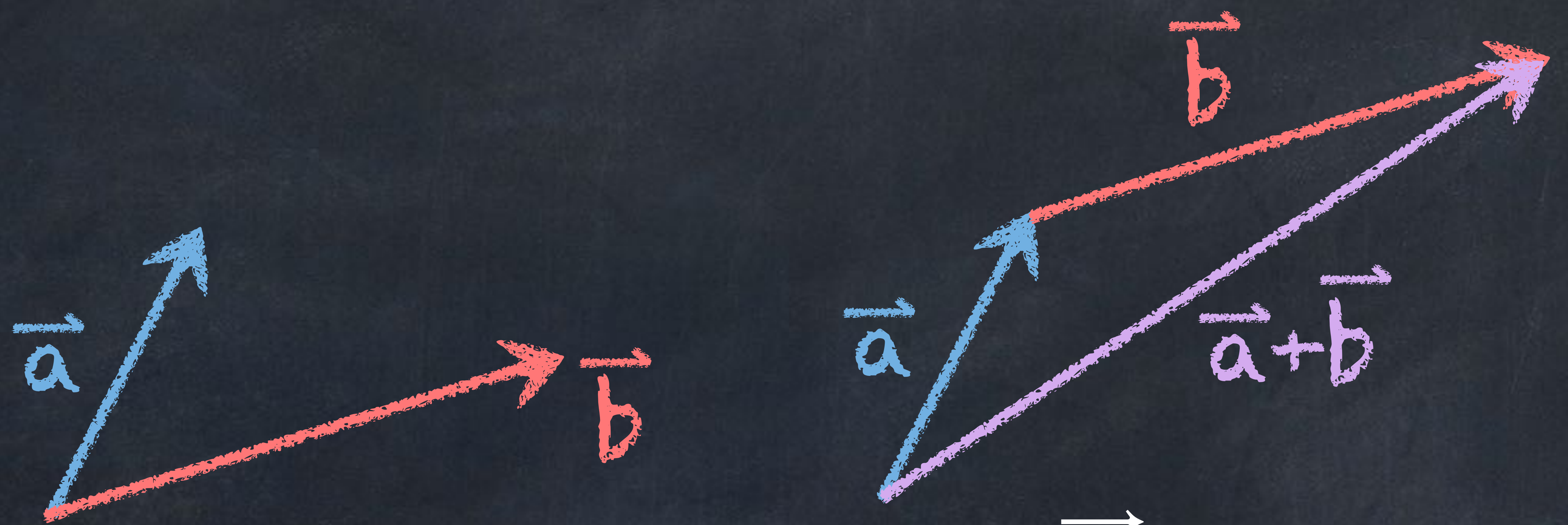
- $\overrightarrow{0} = [0, 0]$ is "the zero vector"
- $\vec{\imath} = [1, 0]$
- $\vec{\jmath} = [0, 1]$

Special vectors in 3D:

- $\overrightarrow{0} = [0, 0, 0]$ is "the zero vector"
- $\vec{\imath} = [1, 0, 0]$
- $\vec{\jmath} = [0, 1, 0]$
- $\overrightarrow{k} = [0, 0, 1]$

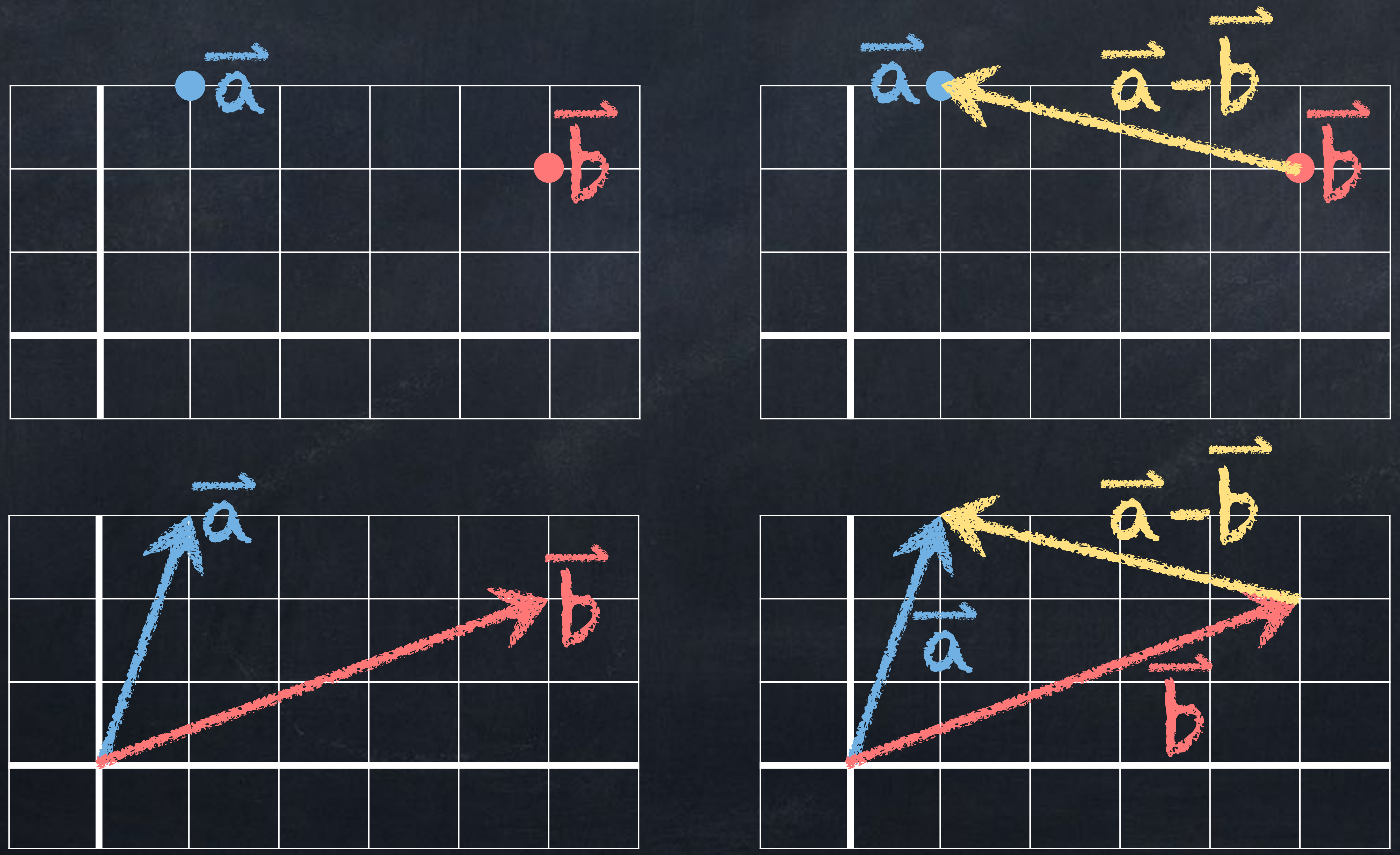
Last week: + and −

- In terms of arrows, we have “tip-to-tail addition”. Example:



- If $\overrightarrow{a}$, $\overrightarrow{b}$ start at the same point, then $\overrightarrow{a} - \overrightarrow{b}$ points from the end of $\overrightarrow{b}$ to the end of $\overrightarrow{a}$. Example:

If $\overrightarrow{a}$ and $\overrightarrow{b}$ start at the origin, then $\overrightarrow{a} - \overrightarrow{b}$ goes from “point $\overrightarrow{b}$” to “point $\overrightarrow{a}$”.



Last week: multiplication

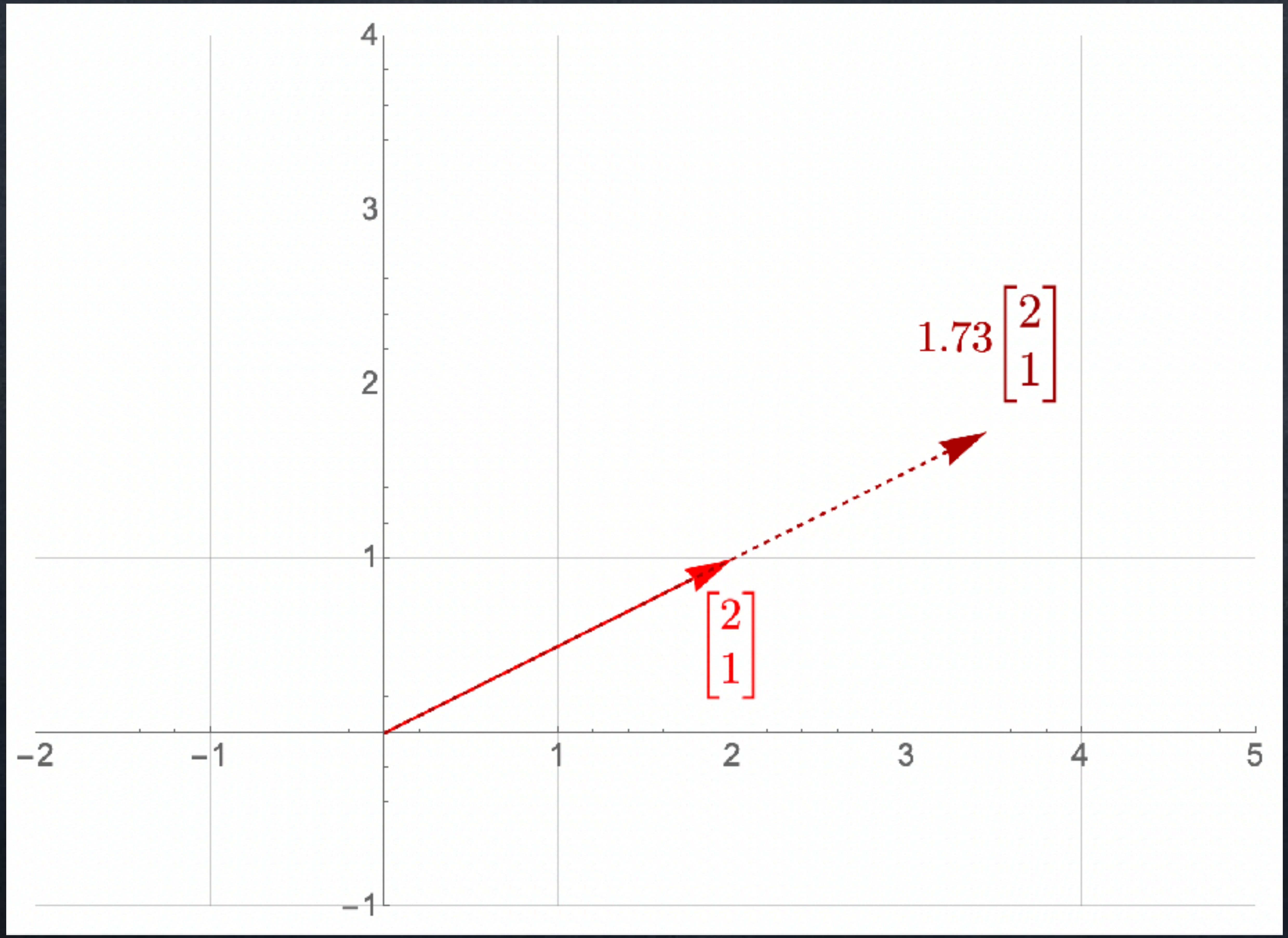
We will *never* combine $\begin{bmatrix}1\\2\\3\end{bmatrix}$ and $\begin{bmatrix}4\\5\\6\end{bmatrix}$ into $\begin{bmatrix}(1)(4)\\(2)(5)\\(3)(6)\end{bmatrix} = \begin{bmatrix}4\\10\\18\end{bmatrix}$ in this class.

Instead we have

- scalar multiplication $7\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}(7)(1)\\(7)(2)\\(7)(3)\end{bmatrix} = \begin{bmatrix}7\\14\\21\end{bmatrix}$
- dot product $\begin{bmatrix}1\\2\\3\end{bmatrix} \cdot \begin{bmatrix}4\\5\\6\end{bmatrix} = (1)(4) + (2)(5) + (3)(6) = 32$
- cross product $\begin{bmatrix}1\\2\\3\end{bmatrix} \times \begin{bmatrix}4\\5\\6\end{bmatrix} = \begin{bmatrix}3\\6\\-3\end{bmatrix}$ to be explained later

Last week: multiplication

A **scalar multiple** of an arrow is a stretched (*scaled*) version—the magnitude can change, but the direction does not.



Last week: magnitude

The **magnitude** (or **length** or **norm**) of the vector $\overrightarrow{v}$ is written $|\overrightarrow{v}|$.

For $\overrightarrow{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$, we have $\left|\overrightarrow{v}\right| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$.

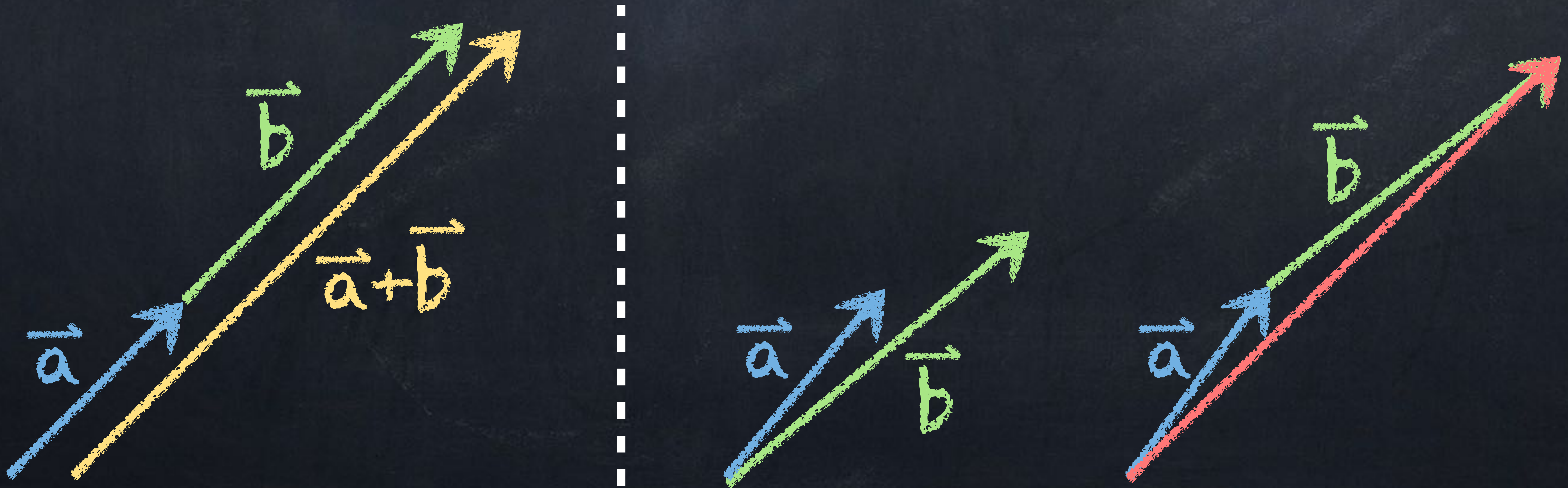
- This is exactly the physical length of the arrow in 2D or 3D.

A vector with magnitude 1 is called a **unit vector**.

- Some people use a hat when writing unit vectors: $\hat{u} = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$.

Properties

- $\left|\vec{a}\right| \geq 0.$
- $\left|\vec{a}\right| = 0$ if and only if $\vec{a} = \vec{0}$.
- $\left|s\vec{a}\right| = |s|\left|\vec{a}\right|.$
- $\left|\vec{a} + \vec{b}\right| \leq \left|\vec{a}\right| + \left|\vec{b}\right|$. This is called the “Triangle Inequality”.



Dot product

The **dot product** of two vectors $\overrightarrow{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ and $\overrightarrow{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, which

we write as $\overrightarrow{a} \cdot \overrightarrow{b}$ (said aloud as "A dot B"), is a *number* that can be computed as either

- $\overrightarrow{a} \cdot \overrightarrow{b} = a_1 b_1 + a_2 b_2$

or

- $\overrightarrow{a} \cdot \overrightarrow{b} = \left|\overrightarrow{a}\right| \left|\overrightarrow{b}\right| \cos\left(\text{angle between } \overrightarrow{a} \text{ and } \overrightarrow{b}\right)$.

Dot product

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Example: Find the acute angle between $\langle 4,3 \rangle$ and $\langle 1,6 \rangle$.

$$|\vec{a}| = \sqrt{16+9} = \sqrt{25} = 5$$

$$|\vec{b}| = \sqrt{1+36} = \sqrt{37}$$

$$\vec{a} \cdot \vec{b} = (4)(1) + (3)(6) = 4 + 18 = 22$$

So $5\sqrt{37}\cos\theta = 22$

$\cos\theta = \dfrac{22}{5\sqrt{37}}$ cannot be solved by hand.

We say $\theta = \arccos\left(\dfrac{22}{5\sqrt{37}}\right)$ or $\cos^{-1}\left(\dfrac{22}{5\sqrt{37}}\right)$.

Orthogonal / perpendicular

Two vectors are called **orthogonal** if their dot product is zero.

- For non-zero vectors, this means they are **perpendicular** (or **normal**).

Why? Using the second dot prod. formula, $\vec{u} \cdot \vec{v} = 0$ means

$$|\vec{u}||\vec{v}|\cos\theta = 0.$$

Since $\vec{u} \neq 0$ and $\vec{v} \neq 0$, $|\vec{u}|$ and $|\vec{v}|$ cannot be 0. So $\cos\theta$ must be 0, and for an acute angle this means θ must be 90°.

- The zero vector is orthogonal to every vector:

$$\begin{aligned}\vec{0} \cdot \langle v_1, v_2, \dots, v_n \rangle &= 0v_1 + 0v_2 + \cdots + 0v_n \\ &= 0 + 0 + \cdots + 0 \\ &= 0\end{aligned}$$

Which of $\overrightarrow{a} = [4, 2]$ or $\overrightarrow{b} = [4, 3]$ is orthogonal to $[2, -4]$?

$\overrightarrow{a}$

Give an example of a non-zero vector that is orthogonal to $\overrightarrow{c} = [3, 7]$.

Any non-zero scalar multiple of [7, -3].
These include [7, -3] and [14, -6]
and [-7, 3] and [-3.5, -1.5].

Set / collection

In mathematics it is often useful to talk about a set (or collection) of objects.

- Usually a set is written using curly brackets { }, and when we use a variable for sets we use capital letters.
  - Example: $S = \{1, 2, 3, 25, 1000\}$.
- Order doesn't matter.
  - $\{38, 4, -5\}$ is exactly the same set as $\{-5, 38, 4\}$.

Some collections have their own special symbols:

- The collection of all natural numbers is written as $\mathbb{N}$ or $\mathbf{N}$.
- The collection of all real numbers is written as $\mathbb{R}$ or $\mathbf{R}$.
- The 2D plane is $\mathbb{R}^2$.
- 3D space is $\mathbb{R}^3$.

Set / collection

- The symbol $\in$ means "is an element of" (or "is in").

- For example since $\mathbb{R}$ is the collection of all real numbers,

$$x \in \mathbb{R}$$

  means "x is in the collection of real numbers", which is the same as "x is a real number".

- More examples:
  - "$k \in \mathbb{N}$" mean "k is a natural number".
  - "$t \in \mathbb{R}$" means "t is a real number".
  - "$\overrightarrow{u} \in \mathbb{R}^3$" means "$\overrightarrow{u}$ is in 3D space".

How to write collections

Instead of listing items, we often describe collections by some rules.

brackets colon brackets

$$\{ \underline{\qquad\qquad} : \underline{\qquad\qquad\qquad\qquad\qquad\qquad} \}$$

kind of object (number, vector, etc.) rules / requirements / descriptions

The following are different ways to write exactly the same statement:

- $S = \{1, 3, 5, 7, 9, \dots\}$
- $S = \{x : x \text{ is an odd natural number}\}$
- $S = \{n \in \mathbb{N} : n \text{ is odd}\}$
- $S = \{2k + 1 : k \in \mathbb{N}\}$

Remember $\in$ means “is in”.

How to write collections

The objects in a collection do not have to be numbers.

Examples:

- $A = \{\langle 4,3\rangle, \langle 2,-5\rangle, \langle 1,31\rangle, \langle -5,9\rangle\}$ is a set of four vectors.
- $B = \{5, 9, 3, 8, 7\}$ is a collection of 5 numbers.
- $C = \{5t : t \in \mathbb{N}\}$ is a set of infinitely many numbers (5, 10, 15, …).
- $D = \{5t : t \in \mathbb{R}\}$ is exactly the collection $\mathbb{R}$.
- $E = \{\langle 4t,10t\rangle : t \in \mathbb{R}\}$ is a collection of vectors that includes $\langle 4,10\rangle$ and $\langle 2,5\rangle$ and $\langle -4, -10\rangle$ (but not $\langle 4,0\rangle$).

How to write collections

The objects in a collection do not have to be numbers.

Examples:

- $A = \left\{ \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \end{bmatrix}, \begin{bmatrix} 1 \\ 31 \end{bmatrix}, \begin{bmatrix} -5 \\ 9 \end{bmatrix} \right\}$ is a set of four vectors.
- $B = \{5, 9, 3, 8, 7\}$ is a collection of 5 numbers.
- $C = \{5t : t \in \mathbb{N}\}$ is a set of infinitely many numbers (5, 10, 15, …).
- $D = \{5t : t \in \mathbb{R}\}$ is exactly the collection $\mathbb{R}$.
- $E = \left\{ \begin{bmatrix} 4t \\ 10t \end{bmatrix} : t \in \mathbb{R} \right\}$ is a collection of vectors that

  includes $\begin{bmatrix} 4 \\ 10 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} -4 \\ -10 \end{bmatrix}$ (but not $\begin{bmatrix} 4 \\ 0 \end{bmatrix}$).

Next week: lines and planes

You will need to be able to work *both* visually *and* with equations/symbols about

- lines in 2D
- lines in 3D
- planes in 3D