Math 1688

9 December 2021

Warm-up:
Subtraction, dot product,
perpendicular vectors.

theadamabrams.com/live

Last week: Lines

A **direction vector** for a line is a vector that is parallel to the line.

If we know

- a point on a line L and
- a direction vector for a line L,

then there is exactly one line L that fits both of these.

$\vec{r}$ means $\begin{bmatrix} x \\ y \end{bmatrix}$ or $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

The line through the point $\overrightarrow{p}$ with direction vector $\overrightarrow{d}$ has equation

$$\vec{r} = \overrightarrow{p} + t\,\overrightarrow{d}$$

in 2D or 3D!

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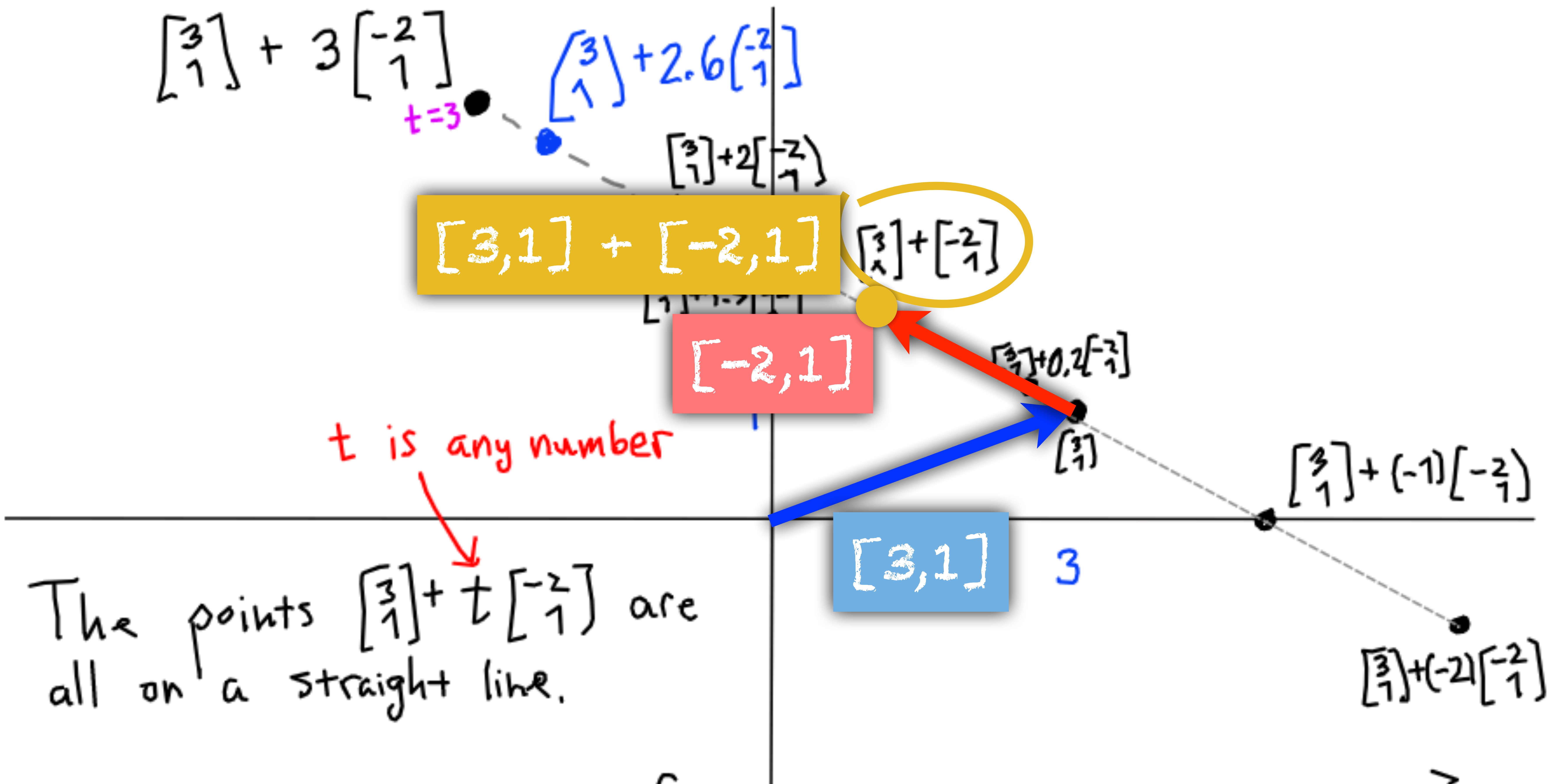
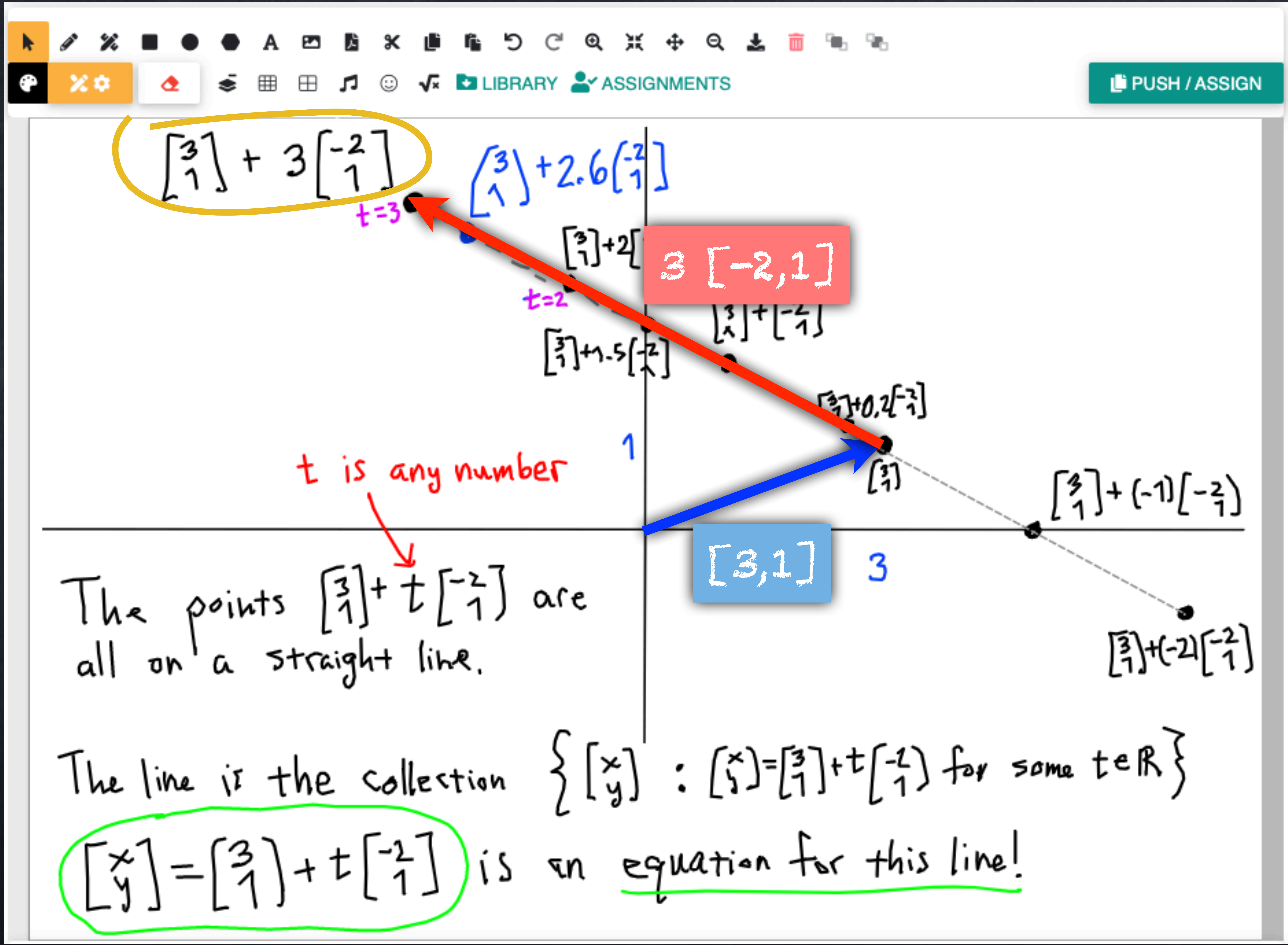


The points $\begin{bmatrix}3\\1\end{bmatrix} + t\begin{bmatrix}-2\\1\end{bmatrix}$ are all on a straight line.

The line is the collection $\left\{ \begin{bmatrix}x\\y\end{bmatrix} : \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}3\\1\end{bmatrix} + t\begin{bmatrix}-2\\1\end{bmatrix} \text{ for some } t \in \mathbb{R} \right\}$

$\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}3\\1\end{bmatrix} + t\begin{bmatrix}-2\\1\end{bmatrix}$ is an equation for this line!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[3,1] + 3[-2,1]
[3,1] + 2.6[-2,1]
t=3
t=2
[3,1]+2[-2,1]
3 [-2,1]
[3,1]+[-2,1]
[3,1]+1.5[-2,1]
[3,1]+0.2[-2,1]
1
t is any number
[3,1]
[3,1]+(-1)[-2,1]
[3,1]
3
[3,1]+(-2)[-2,1]
The points [3,1]+t[-2,1] are all on a straight line.
The line is the collection {[x,y] : [x,y]=[3,1]+t[-2,1] for some t∈R}
[x,y]=[3,1]+t[-2,1] is an equation for this line!

Lines

A **direction vector** for a line is a vector that is parallel to the line.

$\vec{r}$ means $\begin{bmatrix} x \\ y \end{bmatrix}$ or $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

If we know

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in 2D or 3D!

Lines

The line in 2D through the point (x_0, y_0) with direction vector $\overrightarrow{d} = [a, b]$ has parametric equations

$$\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \end{aligned}.$$

The line in 3D through the point (x_0, y_0, z_0) with direction vector $\overrightarrow{d} = [a, b, c]$ has parametric equations

$$\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct \end{aligned}.$$

Give an equation for the line L_1 that goes through the point $(6, 2, 1)$ and is parallel to the line

$$L_2: \quad x = 4 + 2t, \;\; y = -1 + 4t, \;\; z = 5 - t.$$

The direction vector for L_2 is [2,4,-1] (because this is what is multiplied by t). That vector is parallel to L_2, and we want L_1 to be parallel to L_2, so we want [2,4,-1] to be parallel to L_1.

With $\vec{p}$=[6,2,1] and $\vec{d}$=[2,4,-1], we get

$$\vec{r} = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix}$$

or

$$x = 6 + 2t$$
$$y = 2 + 4t$$
$$z = 1 - t$$

Line through point $\vec{p}$ with direction vector $\vec{d}$ has equation $\vec{r} = \vec{p} + t\,\vec{d}$.

Planes

It is possible to make parametric equations for planes, but it requires two parameters.

- Example: The plane through $(-2, 7, 6)$ parallel to both $\begin{bmatrix} 1 \\ 4 \\ 10 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 3 \\ 7 \end{bmatrix}$ has parametric equations

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \\ 6 \end{bmatrix} + t \begin{bmatrix} 1 \\ 4 \\ 10 \end{bmatrix} + s \begin{bmatrix} 3 \\ 3 \\ 7 \end{bmatrix},$$

which we can also write as

$$\begin{aligned} x &= -2 + t + 3s \\ y &= 7 + 4t + 3s \\ z &= 6 + 10t + 7s \end{aligned} \quad .$$

Planes

It is possible to make parametric equations for planes, but it requires two parameters.

However, an equation with only x, y, z (no t or s) will actually be simpler.

Planes

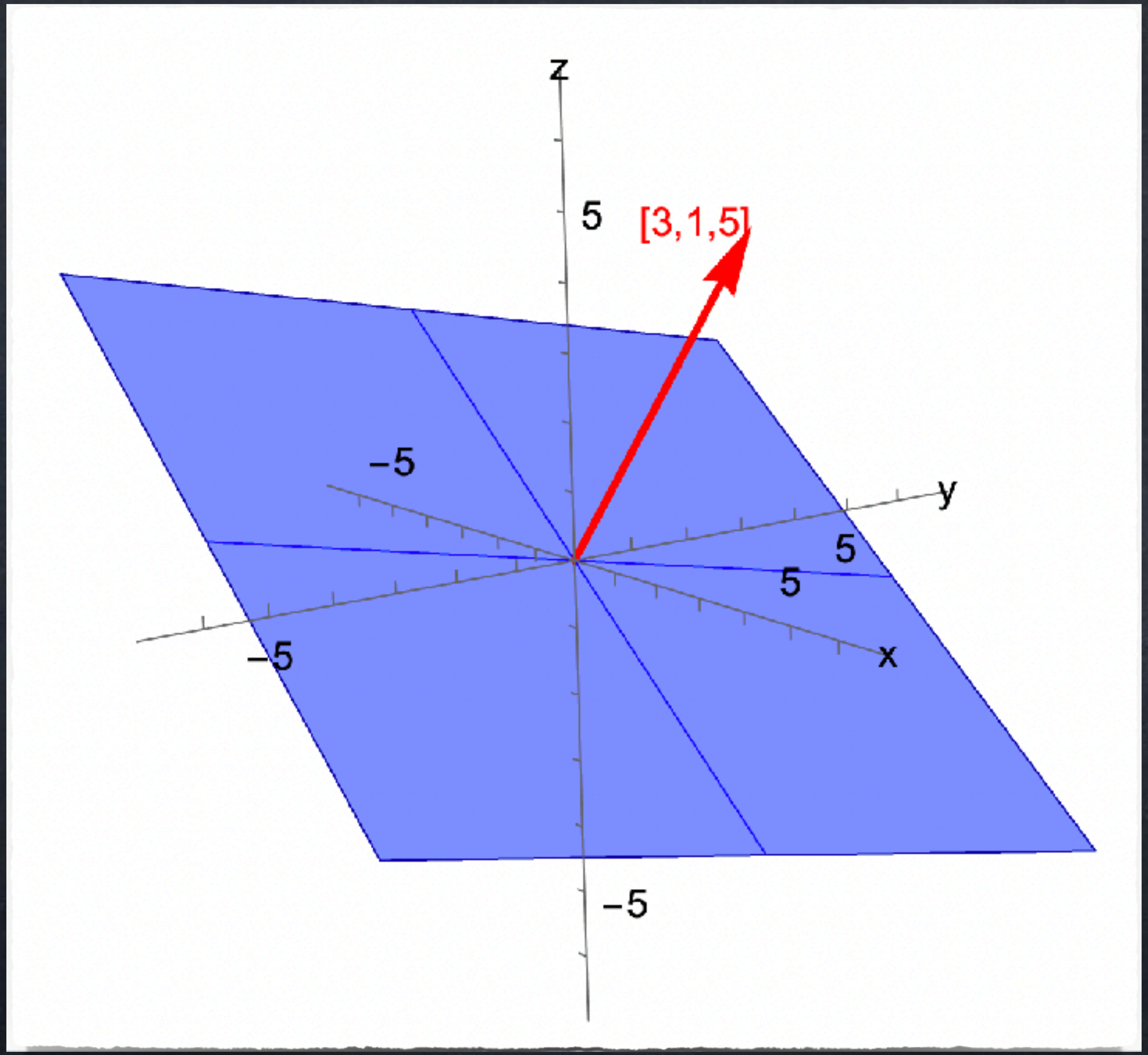
A **normal vector** for a plane is a vector that is perpendicular to the plane.

If we know

- a point on a plane P and
- a normal vector for a plane P,

then there is exactly one plane P that fits both of these.

The plane through $(0,0,0)$ with normal vector $[3, 1, 5]$ looks like



Is $(-1, 2, 8)$ is on the plane through the origin with normal vector $[3, 1, 5]$?

IF the point $(-1, 2, 8)$ is on the plane through $(0,0,0)$ with normal vector $[3, 1, 5]$ then

- the arrow from $(0,0,0)$ to $(-1, 2, 8)$ is perpendicular to $[3, 1, 5]$.
- the vector $[-1, 2, 8]$ is perpendicular to $[3, 1, 5]$.
- the dot product $[3, 1, 5] \cdot [-1, 2, 8]$ equals 0.
- $3(-1) + 1(2) + 5(8) = 0$.
- $39 = 0$.

So we know that $(-1, 2, 8)$ is *not* on this plane.

IF the point $(4, -7, -1)$ is on the plane through $(0,0,0)$ with normal vector $[3, 1, 5]$ then

- the arrow from $(0,0,0)$ to $(4, -7, -1)$ is perpendicular to $[3, 1, 5]$.
- the vector $[4, -7, -1]$ is perpendicular to $[3, 1, 5]$.
- the dot product $[3, 1, 5] \cdot [4, -7, -1]$ equals 0.
- $3(4) + 1(-7) + 5(-1) = 0$.
- $0 = 0$.

So we know that $(4, -7, -1)$ *is* on this plane.

If the point (x, y, z) is on the plane through $(0,0,0)$ with normal vector $[3, 1, 5]$ then

- the arrow from $(0,0,0)$ to (x, y, z) is perpendicular to $[3, 1, 5]$.
- the vector $[x, y, z]$ is perpendicular to $[3, 1, 5]$.
- the dot product $[3, 1, 5] \cdot [x, y, z]$ equals 0.
- $3x + y + 5z = 0.$

So " $3x + y + 5z = 0$ " is the equation for the plane through the origin normal to $[3, 1, 5]$!

$\vec{r}$ means $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

The plane in 3D through $(0, 0, 0)$ with normal vector $\overrightarrow{n}$ has equation

$$\overrightarrow{n} \cdot \vec{r} = 0.$$

If $\overrightarrow{n}$ is [a,b,c], this eqn is ax+by+cz=0.

What if the plane does not go through $(0, 0, 0)$?

Is $(-9, 8, 2)$ is on the plane through $(4, 1, 1)$ with $\overrightarrow{n} = [3, 1, 5]$?

IF it is, then

- the arrow from $(4, 1, 1)$ to $(-9, 8, 2)$ is perpendicular to $[3, 1, 5]$.
- the vector $\left([-9, 8, 2] - [4, 1, 1]\right)$ is perpendicular to $[3, 1, 5]$.
- the dot product $[3, 1, 5] \cdot \left([-9, 8, 2] - [4, 1, 1]\right)$ equals 0.
- $3(-9 - 4) + 1(8 - 1) + 5(2 - 1) = 0.$

What if the plane does not go through $(0, 0, 0)$?

Is $(-9, 8, 2)$ is on the plane through $(4, 1, 1)$ with $\overrightarrow{n} = [3, 1, 5]$?

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- the dot product $[3, 1, 5] \cdot \left([-9, 8, 2] - [4, 1, 1]\right)$ equals 0.
- $3(-9 - 4) + 1(8 - 1) + 5(2 - 1) = 0.$

What if the plane does not go through $(0, 0, 0)$?

Is (x, y, z) is on the plane through $(4, 1, 1)$ with $\overrightarrow{n} = [3, 1, 5]$?

IF it is, then

- the arrow from $(4, 1, 1)$ to (x, y, z) is perpendicular to $[3, 1, 5]$.
- the vector $\big([x, y, z] - [4, 1, 1]\big)$ is perpendicular to $[3, 1, 5]$.
- the dot product $[3, 1, 5] \cdot \big([x, y, z] - [4, 1, 1]\big)$ equals 0.
- $3(x - 4) + 1(y - 1) + 5(z - 1) = 0$.

Planes

A **normal vector** for a plane is a vector that is perpendicular to the plane.

vector form

The plane through the point $\overrightarrow{p}$ with normal vector $\overrightarrow{n}$ has equation

$$\overrightarrow{n} \cdot (\vec{r} - \overrightarrow{p}) = 0.$$

scalar form

The plane through point (x_0, y_0, z_0) with normal vector $[a, b, c]$ has equation

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

Give an equation *without vectors* for the plane through $(12, 4, -3)$ normal to $[-2, 8, 8]$.

With vectors,

$$\begin{bmatrix} -2 \\ 8 \\ 8 \end{bmatrix} \cdot \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 12 \\ 4 \\ -3 \end{bmatrix} \right) = 0$$

Without vectors,

$$-2(x - 12) + 8(y - 4) + 8(z + 3) = 0$$

or

$$-2x + 8y + 8z = -16$$

or

$$x - 4y - 4z = 8$$

Give an equation for the plane through the origin perpendicular to the line

$$L: \quad \vec{r} = \begin{bmatrix} -2 + 10t \\ 9 + t \\ -4 - 2t \end{bmatrix}.$$

The direction vector for the line is [10,1,-2].

That vector is parallel to the line, and the line is perpendicular to our plane, so we can use [10,1,-2] as the normal vector for our plane.

$[10,1,-2] \cdot \vec{r} = 0$ or $\boxed{10x + y - 2z = 0}$

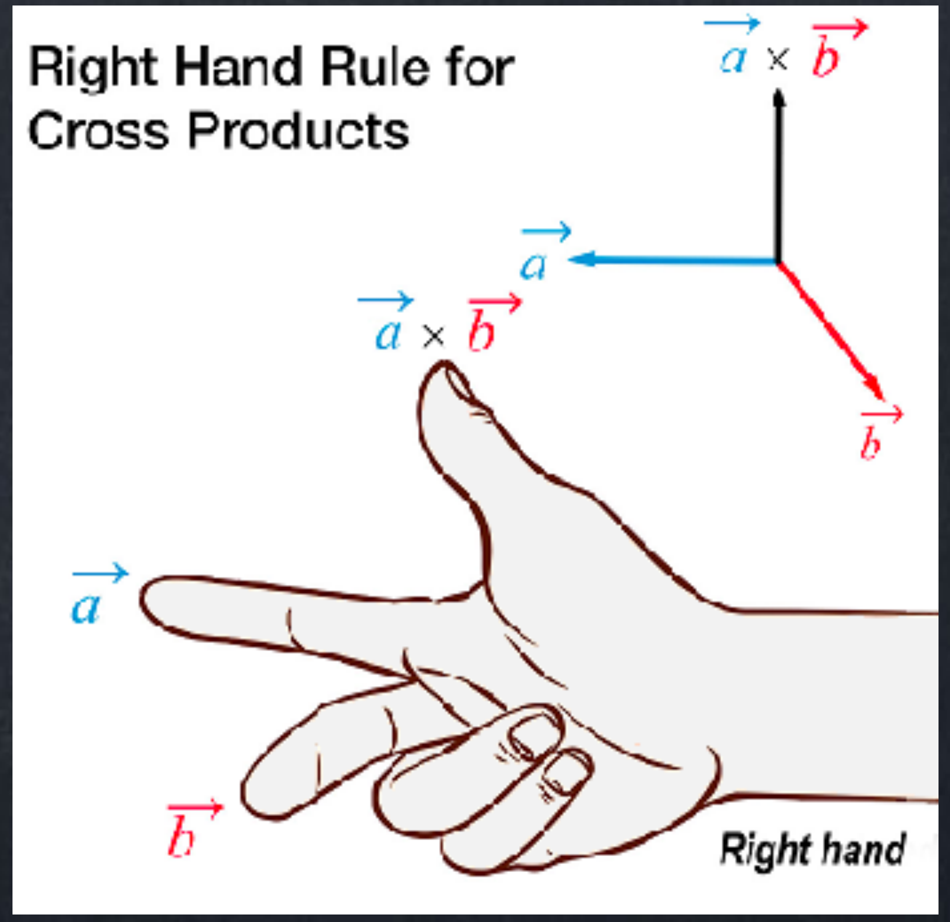
If we wanted an equation $\_\_x + \_\_y + \_\_z = \_\_$ for the plane through $(-2, 7, 6)$ parallel to both $[1, 4, 10]$ and $[3, 3, 7]$, we would need

- a point on the plane use (-2, 7, 6)
- a normal vector for the plane. ??
  - This vector will be perpendicular to both $[1, 4, 10]$ and $[3, 3, 7]$.

Cross product

For 3D vectors only, the **cross product of $\vec{a}$ and $\vec{b}$**, written $\vec{a} \times \vec{b}$, is the vector that is

- perpendicular to both $\vec{a}$ and $\vec{b}$ and
- has length $|\vec{a}|\,|\vec{b}|\,\sin(\text{angle between } \vec{a} \text{ and } \vec{b})$ and
- follows the "Right-Hand Rule".



The formula is ugly:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}.$$

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Linear combinations

A **linear combination** of some vectors is any sum of scalar multiples of those vectors.

- In symbols, $\overrightarrow{u}$ is a linear combination of $\overrightarrow{v}$ and $\overrightarrow{w}$ if
$$\overrightarrow{u} = s\overrightarrow{v} + t\overrightarrow{w}$$
for some numbers s, t.

- For more vectors, $\overrightarrow{u}$ is a linear combination of $\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots, \overrightarrow{v_n}$ if
$$\overrightarrow{u} = s_1\overrightarrow{v_1} + s_2\overrightarrow{v_2} + \cdots + s_n\overrightarrow{v_n}$$
for some numbers $s_1, \ldots, s_n$.

Linear combinations

A **linear combination** of some vectors is any sum of scalar multiples of those vectors.

- In symbols, $\overrightarrow{u}$ is a linear combination of $\overrightarrow{v}$ and $\overrightarrow{w}$ if

$$\overrightarrow{u} = s\overrightarrow{v} + t\overrightarrow{w}$$

for some numbers s, t.

Example 1: Write $\begin{bmatrix} 5 \\ 24 \end{bmatrix}$ as a linear combination of $\overrightarrow{v_1} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and $\overrightarrow{v_2} = \begin{bmatrix} 0 \\ 8 \end{bmatrix}$.

$s\begin{bmatrix} 1 \\ 4 \end{bmatrix} + t\begin{bmatrix} 0 \\ 8 \end{bmatrix} = \begin{bmatrix} s \\ 4s+8t \end{bmatrix}$. For this to equal $\begin{bmatrix} 5 \\ 24 \end{bmatrix}$ we need $s = 5$.

Then $4(5) + 8t = 24$, so $8t = 4$, so $t = 1/2$.

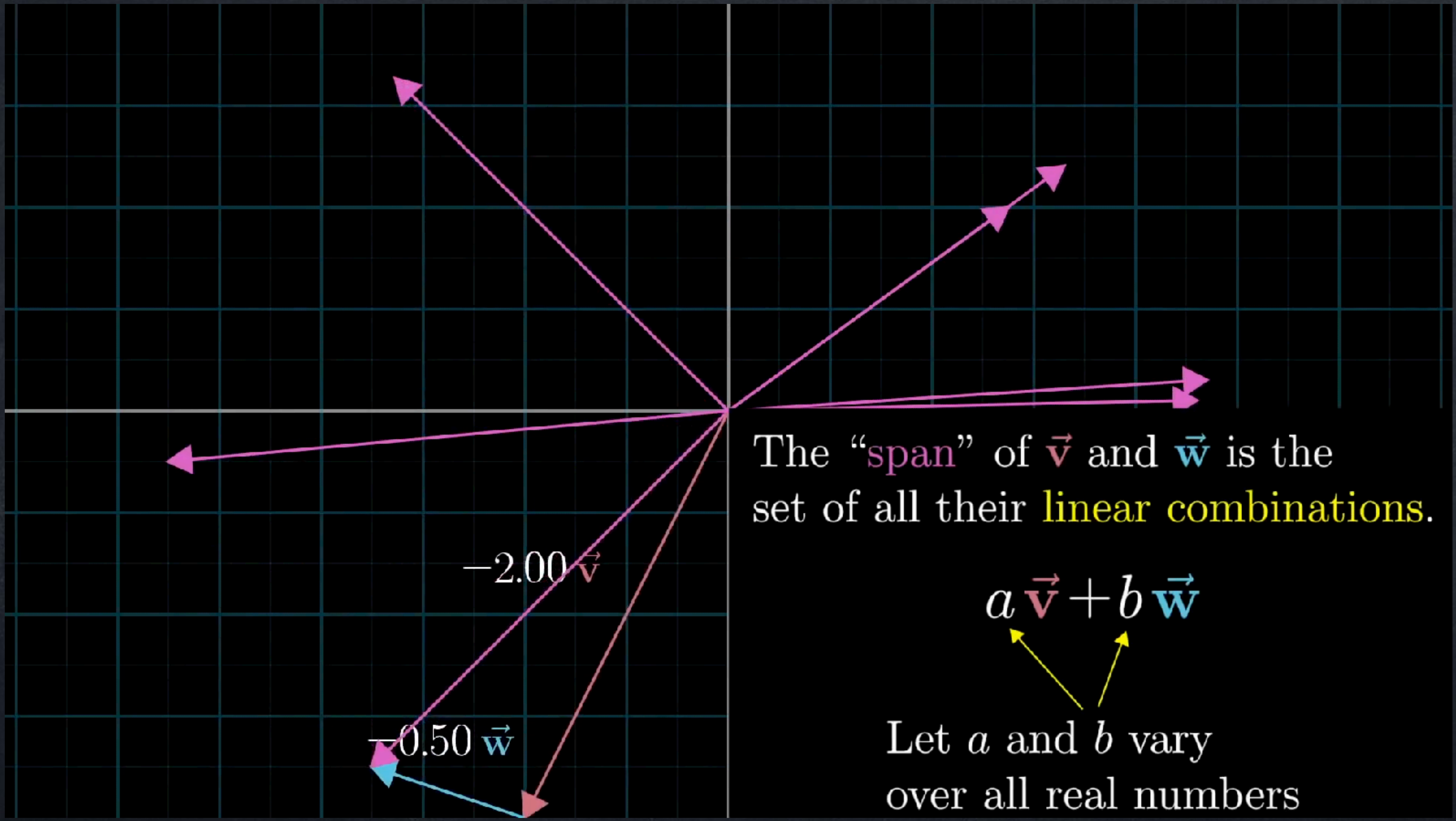
$$5\begin{bmatrix} 1 \\ 4 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 \\ 8 \end{bmatrix}$$

Example 2: $\begin{bmatrix} 5 \\ 24 \end{bmatrix}$ *cannot* be written as a linear combination of $\overrightarrow{v_1} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ and $\overrightarrow{v_2} = \begin{bmatrix} 10 \\ 2 \end{bmatrix}$. Why?

Using equations:

$$s\begin{bmatrix} 5 \\ 1 \end{bmatrix} + t\begin{bmatrix} 10 \\ 2 \end{bmatrix} = \begin{bmatrix} 5s+10t \\ s+2t \end{bmatrix} = \begin{bmatrix} 5(s+2t) \\ s+2t \end{bmatrix} = (s+2t)\begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

To get "$n\begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 24 \end{bmatrix}$" would require $n=1$ and $n=3$ at the same, which is impossible.



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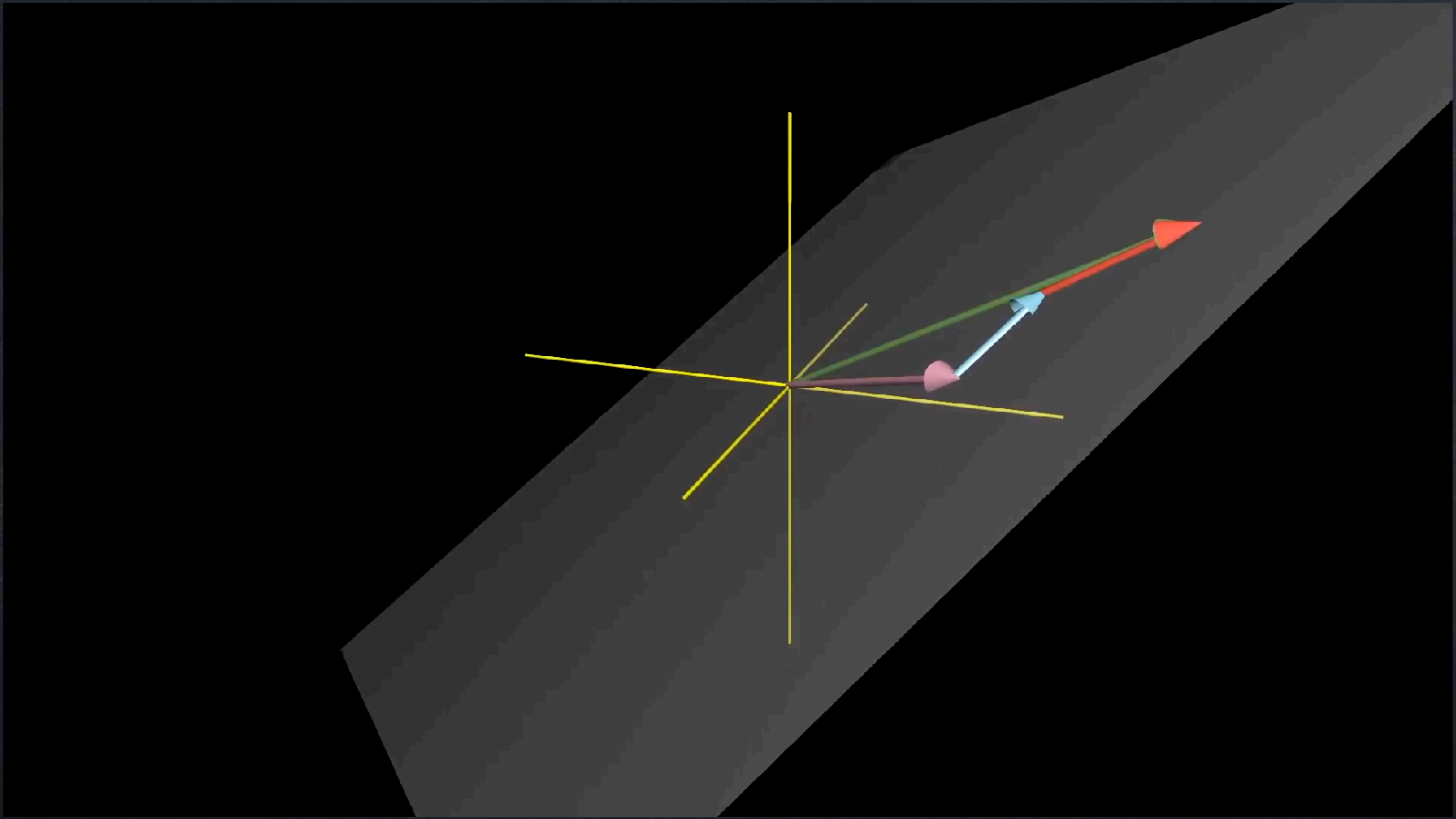

from 3Blue1Brown — youtu.be/k7RM-ot2NWY

Span

The **span** of some vectors is the collection of all linear combinations of the vectors.

- For two vectors,

$$\text{span}(\vec{v}, \vec{w}) = \left\{ s\vec{v} + t\vec{w} \ : \ s, t \in \mathbb{R} \right\}$$

Earlier we saw that $\begin{bmatrix} 5 \\ 24 \end{bmatrix}$ cannot be written as a linear combination of $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 10 \\ 2 \end{bmatrix}$. We can say "$\begin{bmatrix} 5 \\ 24 \end{bmatrix}$ is not in the span of $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 10 \\ 2 \end{bmatrix}$."

Span

The **span** of some vectors is the collection of all linear combinations of the vectors.

- For two vectors,

$$\operatorname{span}(\overrightarrow{v}, \overrightarrow{w}) = \left\{ s\overrightarrow{v} + t\overrightarrow{w} \ : \ s, t \in \mathbb{R} \right\}$$

  is usually a plane, but it might be a line or just the origin.

- For three vectors, what shape is

$$\operatorname{span}(\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}) = \left\{ r\overrightarrow{u} + s\overrightarrow{v} + t\overrightarrow{w} \ : \ r, s, t \in \mathbb{R} \right\}?$$

  - If $\overrightarrow{u}, \ldots$ are 2D vectors, this can still be only a point, line, or plane.
  - If $\overrightarrow{u}, \ldots$ are 3D vectors, this can be a point, line, plane, or 3D space.