Linear Algebra, Winter 2021

List 1

Algebra/trig review, complex numbers.

1. Classify each of the following as an "expression", "equation", or "inequality".
   (a) $5x^2 + 2$ expression
   (b) $8 = 9$ equation
   (c) $3x^5 - \sqrt{x}$ expression
   (d) $\sin(\pi)$ expression
   (e) $9x^3 - 5 + i \leq 0$ inequality
   (f) $9x^3 - 5 + i > 0$ inequality
   (g) $9x^3 - 5 + i = 0$ equation
   (h) $4^x = 2x - 17$ equation

2. Which of the following are true for **all** real values of the variables?
   (a) $2x = x + x$ True
   (b) $2(x + y) = 2x + y$ False
   (c) $(x - y)^2 = x^2 - 2xy + y^2$ True
   (d) $(6 + a)/2 = 3 + a/2$ True
   (e) $-(y + 2) = -y + 2$ False
   (f) $-(a + b)^2 = (-a + b)^2$ False
   (g) $x^3 + 3x = x + x$ False
   (h) $k^{-2} = 1/k^2$ True
   (i) $x^{a+2} = x^a \times x^2$ True

3. Compute the following values:
   (a) $\cos(0)$ 1
   (b) $\sin(0)$ 0
   (c) $\cos(30°)$ $\sqrt{3}/2$
   (d) $\cos(45°)$ $1/\sqrt{2}$ or $\sqrt{2}/2$
   (e) $\cos(60°)$ $1/2$
   (f) $\cos(\pi/3)$ $1/2$ (same as previous)
   (g) $\cos(\pi/2)$ 0
   (h) $\sin(\pi/2)$ 1

(i) $\sin(120°)$ $\boxed{\sqrt{3}/2}$

(j) $\sin(5\pi/6)$ $\boxed{1/2}$

(k) $\sin(180°)$ $\boxed{0}$

(ℓ) $\sin(4\pi/3)$ $\boxed{-\sqrt{3}/2}$

(m) $\cos(315°)$ $\boxed{1/\sqrt{2} \text{ or } \sqrt{2}/2}$

(n) $\cos(-45°)$ $\boxed{1/\sqrt{2} \text{ or } \sqrt{2}/2}$ (same as previous)

(o) $\cos(675°)$ $\boxed{1/\sqrt{2} \text{ or } \sqrt{2}/2}$ (same as previous)

(p) $\arccos(\frac{\sqrt{3}}{2})$ $\boxed{\pi/6 \text{ or } 30°}$

The number "i" satisfies $i \times i = -1$. If a and b are any real numbers (including zero), then the number "$a + bi$" is called a **complex number**. The **real part** of this number is a, the **imaginary part** of this number is b, and the **conjugate** of this number is $a - bi$.

For a complex number z, we write $\text{Re}(z)$ for its real part, $\text{Im}(z)$ for its imaginary part, and $\overline{z}$ (spoken as "z bar") for its conjugate.

4. Write the following in rectangular form $a + bi$ (also called Cartesian form):

(a) $(-6 + 5i) + (2 - 4i)$ $\boxed{-4 + i}$

(b) $(1 + 2i)(2 + 3i)$ $\boxed{-4 + 7i}$

(c) $(-5 + 2i) - (2 - i)$ $\boxed{-7 + 3i}$

(d) $(2 - 3i)(2 + 3i)$ $\boxed{13}$

(e) $(1 + i)(2 - i)(3 + 2i)$ $\boxed{7 + 9i}$

(f) $(1 - 2i)^3$ $\boxed{-11 + 2i}$

(g) $(-2i)^6$ $\boxed{-64}$

(h) $\overline{5 + 6i}$ $\boxed{5 - 6i}$

(i) $\overline{-1 - 9i}$ $\boxed{-1 + 9i}$

(j) $\overline{3i}$ $\boxed{-3i}$

(k) $\overline{12}$ $\boxed{12}$

(ℓ) $(2 - 3i)(\overline{2 - 3i})$ $\boxed{13}$

(m) $\text{Re}(2i - 7)$ $\boxed{-7}$ because $2i - 7 = (-7) + (2)i$

(n) $\text{Im}\big((3 + 2i)(5i)\big)$ $\boxed{15}$ because $(3 + 2i)(5i) = -10 + 15i$

(o) $\text{Re}(i^2)$ $\boxed{-1}$ because $i^2 = -1 = (-1) + (0)i$

5. Write $\dfrac{1+2i}{2-3i}$ in the form $a+bi$.

Using 6(b) and 6(ℓ), $\dfrac{(1+2i)(2+3i)}{(2-3i)(2+3i)} = \dfrac{-4+7i}{13} = \boxed{\dfrac{-4}{13} + \dfrac{7}{13}i}$

The **magnitude** (or **modulus**) of $z = a+bi$, written $|z|$, is the distance between z and 0 on the complex plane (that is, between $(0,0)$ and (a,b) on an xy-plane). The **argument** of z, written $\arg(z)$, is the angle between the positive real axis and the line from 0 to z.

We write $r \cdot e^{(\theta \cdot i)}$ for the complex number $r\cos(\theta) + r\sin(\theta)\,i$.
Notice that $re^{\theta i}$ has magnitude r and argument θ.

6. Compute $|1-\sqrt{3}i|$ and $\arg(1-\sqrt{3}i)$.

7. Write the following numbers in polar form $re^{i\theta}$ (also called trigonometric form):

   (a) $-3i$ $\boxed{3e^{-i\pi/2} \text{ or } -3e^{i\pi/2}}$

   (b) $1+\sqrt{3}i$ $\boxed{2e^{i\pi/3}}$

   (c) $2-2\sqrt{3}i$ $\boxed{4e^{(-\pi/3)i} \text{ or } 4e^{(5\pi/3)i}}$

   (d) $\dfrac{\sqrt{3}-i}{7}$ $\boxed{\frac{2}{7}e^{(-\pi/6)i}}$

   (e) $\cos(\frac{5}{11}\pi) + i\sin(\frac{5}{11}\pi)$ $\boxed{e^{(5\pi/11)i}}$

   (f) $\sqrt{-1}$ $\boxed{e^{\frac{\pi}{2}i}}$

8. Write the following in rectangular form:

   (a) $e^{\frac{\pi}{4}i}$ $\boxed{\dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}i \text{ or } \dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2}i}$

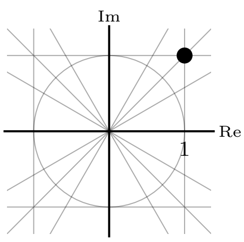
   (b) $2e^{i\pi/6}$ $\boxed{\sqrt{3}+i}$

   (c) $5e^{-i\pi/3}$ $\boxed{\dfrac{5}{2} - \dfrac{5\sqrt{3}}{2}i}$

   (d) $-8e^{\pi i}$ $\boxed{8 \text{ or } 8+0i}$

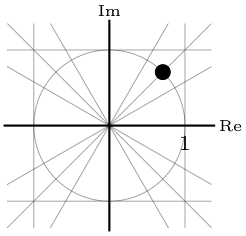
   (e) $\sqrt{9}+\sqrt{-9}$ $\boxed{3+3i}$

9. Write each number in both rectangular and polar form:

   (a) 

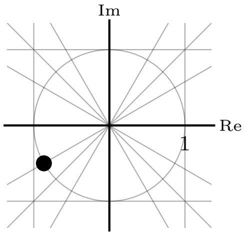
$\boxed{1+i = \sqrt{2}e^{i\pi/4}}$

(b)



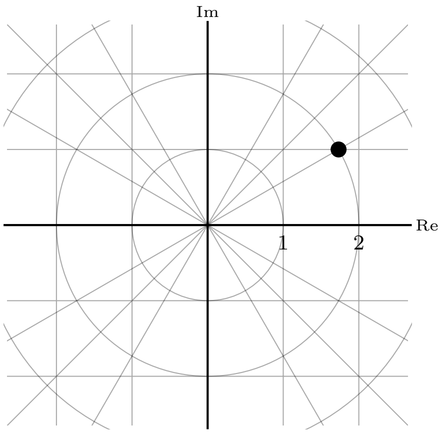
$\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = e^{i\pi/4}$

(c)



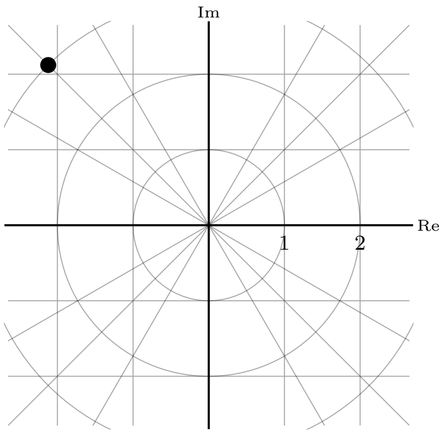
$e^{(7\pi/6)i} = e^{(-5\pi/6)i} = -\frac{\sqrt{3}}{2} - \frac{1}{2}i$

(d)



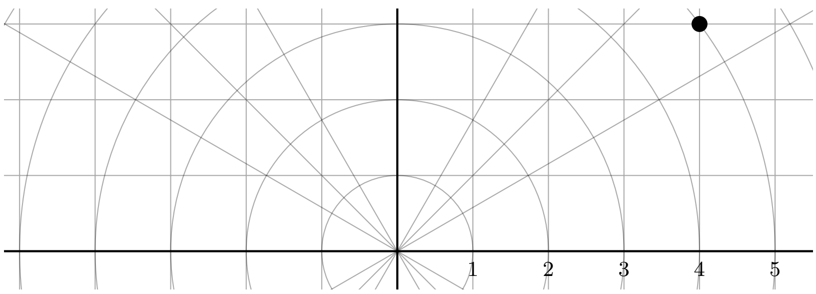
$2e^{(\pi/6)i} = \sqrt{3} + i$

(e)



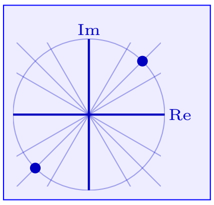
$3e^{(3\pi/4)i} = \frac{3}{\sqrt{2}} + \frac{-3}{\sqrt{2}}i$

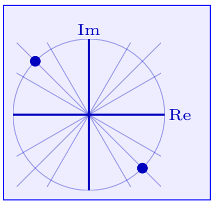
(f)

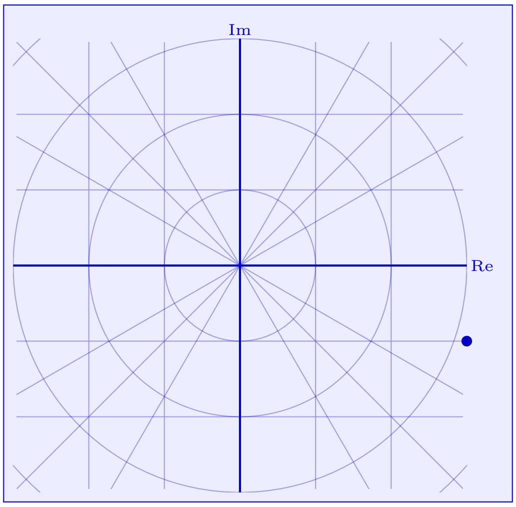


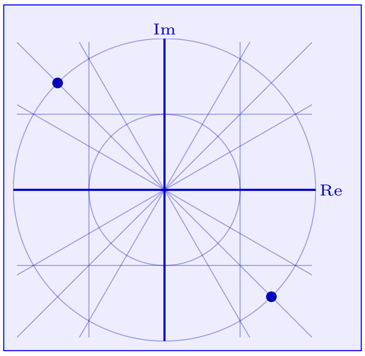
$4 + 3i = 5e^{\arctan(3/4)i}$

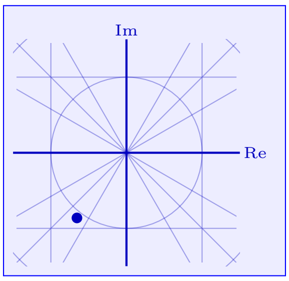
10. On a complex plane, draw the number(s)...

(a) $\dfrac{1+i}{\sqrt{2}}$ and $\dfrac{-1-i}{\sqrt{2}}$ 

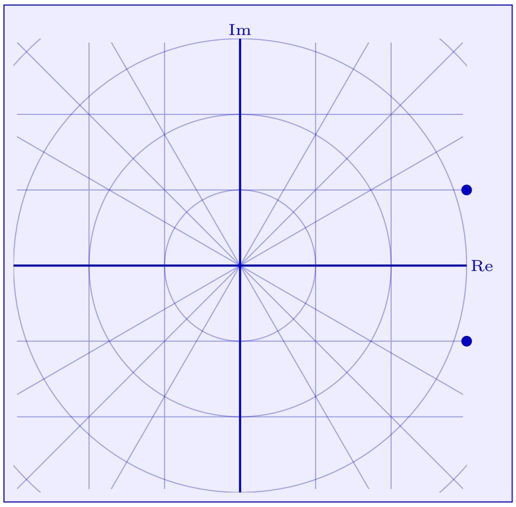
(b) $\dfrac{1-i}{\sqrt{2}}$ and $\dfrac{-1+i}{\sqrt{2}}$ 

(c) $3-i$ 

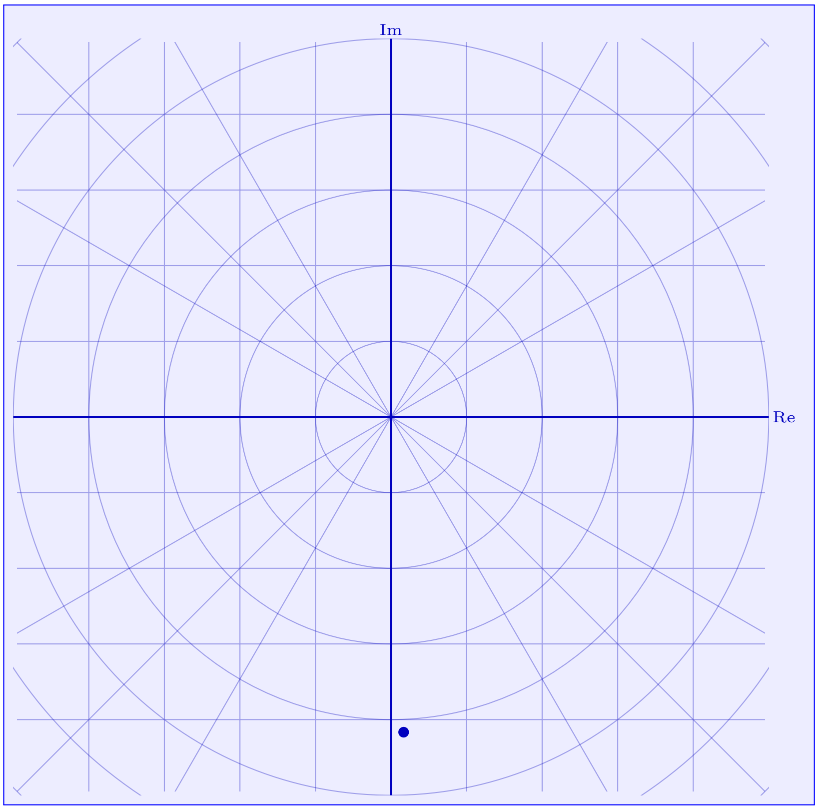
(d) $\sqrt{2}-\sqrt{2}i$ and $-\sqrt{2}+\sqrt{2}i$ 

(e) $-\dfrac{19}{29}-\dfrac{25}{29}i$ 

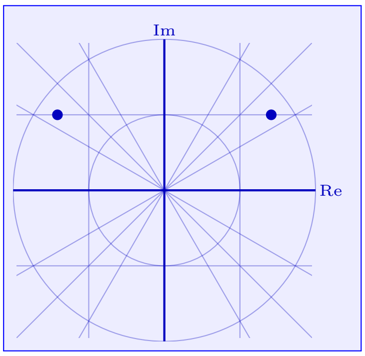
(f) $3 - i$ and $3 + i$



(g) $\dfrac{1 - 25i}{6}$



(h) $i - \sqrt{2}$ and $i + \sqrt{2}$



11. Compute the magnitude (also called modulus) of the number...

(a) $2 + 7i$ $\boxed{\sqrt{53}}$

(b) $\dfrac{4+i}{3+2i}$ $\boxed{\sqrt{17/13}}$

(c) $(1+\sqrt{2}i)$ $\boxed{\sqrt{3}}$

(d) $\dfrac{(3-i\sqrt{3})^2}{(\sqrt{2}+2i)^3}$ $\boxed{\sqrt{2/3}}$