Linear Algebra, Winter 2021

List 3

Polynomials

The number c is a **zero** (also called a **root**) of the polynomial $f(z)$ if $f(c) = 0$.

23. For the polynomial

$$f(z) = z^4 + z^3 + z^2 + 3z - 6 = (z^2 + z - 2)(z^2 + 3),$$

(a) find all roots of $f(z)$. $\boxed{-2, 1, i\sqrt{3}, -i\sqrt{3}}$

(b) find all zeroes of $f(z)$. (same as #23(a))

(c) solve $f(z) = 0$ for complex z. $\boxed{z = -2, z = 1, z = \sqrt{3}i, z = -\sqrt{3}i}$

(d) factor $f(z)$ into irreducible complex factors. $\boxed{(z+2)(z-1)(z-\sqrt{3}i)(z+\sqrt{3}i)}$

(e) list all polynomials of the form $z +$ ____ for which the remainder when dividing $f(z)$ by these polynomials is 0. (same as #23(d))

24. Given that 4 is one root of $z^3 - 4z^2 + 49z - 196$, find all the roots of this polynomial. $\boxed{4, 7i, -7i}$

25. Given that $3 - i$ is one root of $z^3 + 2z^2 - 38z + 80$, find all the roots of this polynomial. $\boxed{3 - i, 3 + i, -8}$

26. Given that $1 + 2i$ is one root of $z^4 - 4z^3 + 12z^2 - 16z + 15$, find all the roots of this polynomial.

$1 - 2i$ must also be a root.

$\dfrac{z^4 - 4z^3 + 12z^2 - 16z + 15}{(z - (1+2i))(z - (1-2i))}$ must be a polynomial.

$\dfrac{z^4 - 4z^3 + 12z^2 - 16z + 15}{z^2 - 2z + 5}$ must be a polynomial.

Long division:

$$\begin{array}{r|rrrrr}
 & z^2 & -2z & +3 & & \\ \hline
z^2 - 2z + 5 & z^4 & -4z^3 & +12z^2 & -16z & +15 \\
 & -(z^4 & -2z^3 & +5z^2) & & \\ \cline{2-4}
 & & -2z^3 & +7z^2 & & \\
 & & -(-2z^3 & +4z^2 & -10z) & \\ \cline{3-5}
 & & & 3z^2 & -6z & \\
 & & & -(3z^2 & -6z & +15) \\ \cline{4-6}
 & & & & & 0
\end{array}$$

So $\dfrac{z^4 - 4z^3 + 12z^2 - 16z + 15}{z^2 - 2z + 5} = z^2 - 2z + 3$.

By the QF, the roots of $z^2 - 2z + 3$ are $\dfrac{2 \pm \sqrt{4 - 12}}{2} = 1 \pm i\sqrt{2}$.

The roots of the original polynomial are $\boxed{1 + 2i, 1 - 2i, 1 + i\sqrt{2}, 1 - i\sqrt{2}}$.

For any two polynomials f and g, there exist unique polynomials Q and R such that $f = Q \cdot g + r$ and either $R(x) = 0$ or the degree of R is less than the degree of g. The **quotient** of f divided by g is $Q(x)$, and the **remainder** of f divided by g is $R(x)$.

27. Using the fact that

$$\frac{7x^4 + 2x^3 - 8}{x^2 - 4} = 7x^2 + 2x + 28 + \frac{8x + 104}{x^2 - 4},$$

find polynomials $Q(x)$ and $R(x)$ such that

$$7x^4 + 2x^3 - 8 = (x^2 - 4) \cdot Q(x) + R(x).$$

$\boxed{Q(x) = 7x^2 + 2x + 28, \quad R(x) = 8x + 104}$ are the quotient and remainder. Since the problem did not specify that R must have degree less than the the degree of $x^2 - 4$, there are technically many other correct answers to #27.

28. Find the quotient and remainder when $7x^4 + 2x^3 - 8$ is divided by $x^2 - 4$.

$\boxed{\text{Quotient: } 7x^2 + 2x + 35. \text{ Remainder: } 10x + 167.}$

29. (a) If f is a polynomial of degree 8, and g is a polynomial of degree 5, and g is not a factor of f, then what are the possible values for the degree of the remainder $R(x)$ of f divided by g? $\boxed{0, 1, 2, 3, 4}$

(b) If f is a polynomial of degree 8, and g is a polynomial of degree 1, and g is not a factor of f, then what are the possible values for the degree of the remainder $R(x)$ of f divided by g? $\boxed{0}$

30. Find the remainder when $\frac{1}{12}x^4 + 3x^3 - 7x + 9$ is divided by $x - 2$. $\boxed{\frac{61}{3}}$

31. Find the value of a for which the remainder of $x^3 + (1 - 3a)x^2 + (2 + a^2)x - 20$ divided by $x - 4$ is as small as possible.

$4a^2 - 48a + 68$ is minimal for $\boxed{a = 6}$.

A polynomial is called **irreducible** if it is *not* the product of two non-constant polynomials.

32. Factor the following polynomials into irreducible real factors:

(a) x^3+x^2+x+1 $\boxed{(x + 1)(x^2 + 1)}$

(b) x^3+x^2-x-1 $\boxed{(x + 1)(x + 1)(x - 1) \text{ or } (x + 1)^2(x - 1)}$

(c) x^4-4x^3+8x $\boxed{x(x - 2)(x^2 - 2x - 4)}$

(d) x^4+5x^2+6 $\boxed{(x^2 + 2)(x^2 + 3)}$

33. Factor the following polynomials into irreducible complex factors:

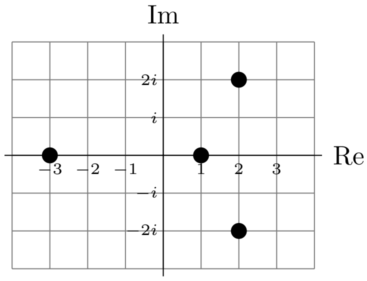
(a) z^3+z^2+z+1 $\boxed{(z + 1)(z + i)(z - i)}$

(b) z^3+z^2-z-1 $(z+1)(z+1)(z-1)$ or $(z+1)^2(z-1)$

(c) z^4-4z^3+8z $z(z-2)(z-\sqrt{5}+1)(z+\sqrt{5}-1)$

(d) z^4+5z^2+6 $(z+i\sqrt{2})(z-i\sqrt{2})(z+i\sqrt{3})(z-i\sqrt{3})$

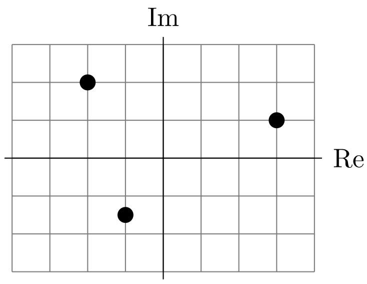
34. The figure below shows four points on the complex plane.



Give an example of a polynomial of degree 4 whose roots are exactly these four points.

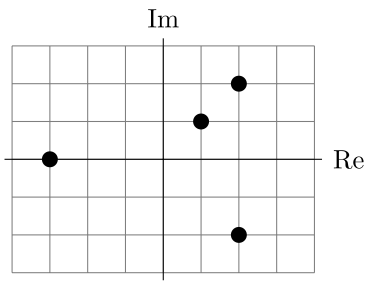
$(z+3)(z-(2+2i))(z-(2-2i))(z-1)$ or $z^4-2z^3-3z^2+28z-24$ or scalar multiples of these (e.g., $100z^4-200z^3-300z^2+2800z-2400$).

35. The figure below shows three points on the complex plane.



For which values of d does there exist a complex polynomial of degree d whose roots are exactly these three points? All $d \geq 3$

36. The figure below shows four points on the complex plane.



(a) Does there exist a polynomial with real coefficients whose roots are exactly these four points? Why or why not? No because if a real polyn. has $1+i$ as a root it must also have $\overline{1+i}=1-i$.

(b) Does there exist a polynomial with complex coefficients whose roots are exactly these four points? Why or why not? Yes: $(z-c_1)\cdots(z-c_4)$ always works.

37. (a) For $f(x) = (z-3)^4(z+2)$, what is the multiplicity of 3? $\boxed{4}$

(b) For $f(x) = z^3 + 2z^2 - 7z + 4$, what is the multiplicity of 1? $\boxed{2}$

38. The only roots of $z^5 - 4z^4 + z^3 + 10z^2 - 4z - 8$ are -1 (with some multiplicity) and $+2$ (with some multiplicity). What is the sum of these multiplicities? $\boxed{5}$

39. The polynomial $f(z) = 3z^6 - 25z^5 + 62z^4 - 132z^3 + 89z^2 - 107z + 30$ has no repeated roots. How many monic linear polynomials are factors of $f(z)$? $\boxed{6}$

40. (a) Write $\dfrac{5x+6}{x^2-6x+8}$ as a sum of partial fractions. $\boxed{\dfrac{13}{x-4} + \dfrac{-8}{x-2}}$

(b) Write $\dfrac{36}{x^3+9x^2+18x}$ as a sum of partial fractions. $\boxed{\dfrac{2}{x+6} + \dfrac{-4}{x+3} + \dfrac{2}{x}}$

41. Given that $2x^3 - 13x^2 + 26x - 10 = (2x-1)(x^2-6x+10)$, write $\dfrac{29}{2x^3-13x^2+26x-10}$ as a sum of partial fractions.

$\boxed{\dfrac{-2x+11}{x^2-6x+10} + \dfrac{4}{2x-1}}$