Linear Algebra, Winter 2021

List 3

Polynomials

The number c is a **zero** (also called a **root**) of the polynomial $f(z)$ if $f(c) = 0$.

23. For the polynomial

$$f(z) \;=\; z^4 + z^3 + z^2 + 3z - 6 \;=\; (z^2 + z - 2)(z^2 + 3),$$

   (a) find all roots of $f(z)$.

   (b) find all zeroes of $f(z)$.

   (c) solve $f(z) = 0$ for complex z.

   (d) factor $f(z)$ into irreducible complex factors.

   (e) list all linear monic polynomials $g(z)$ for which the remainder when dividing $f(z)$ by $g(z)$ is 0.

24. Given that 4 is one root of $z^3 - 4z^2 + 49z - 196$, find all the roots of this polynomial.

25. Given that $3 - i$ is one root of $z^3 + 2z^2 - 38z + 80$, find all the roots of this polynomial.

26. Given that $1 + 2i$ is one root of $z^4 - 4z^3 + 12z^2 - 16z + 15$, find all the roots of this polynomial.

For any two polynomials f and g, there exist unique polynomials Q and R such that $f = Q \cdot g + r$ and either $R(x) = 0$ or the degree of R is less than the degree of g. The **quotient** of f divided by g is $Q(x)$, and the **remainder** of f divided by g is $R(x)$.

27. Using the fact that

$$\frac{7x^4 + 2x^3 - 8}{x^2 - 4} = 7x^2 + 2x + 28 + \frac{8x + 104}{x^2 - 4},$$

   find polynomials $q(x)$ and $r(x)$ such that

$$7x^4 + 2x^3 - 8 = (x^2 - 4) \cdot q(x) + r(x).$$

28. Find the quotient and remainder when $7x^4 + 2x^3 - 8$ is divided by $x^2 - 4$.

29. (a) If f is a polynomial of degree 8, and g is a polynomial of degree 5, and g is not a factor of f, then what are the possible values for the degree of the remainder $R(x)$ of f divided by g?

   (b) If f is a polynomial of degree 8, and g is a polynomial of degree 1, and g is not a factor of f, then what are the possible values for the degree of the remainder $R(x)$ of f divided by g?

30. Find the remainder when $\frac{1}{12}x^4 + 3x^3 - 7x + 9$ is divided by $x - 2$.

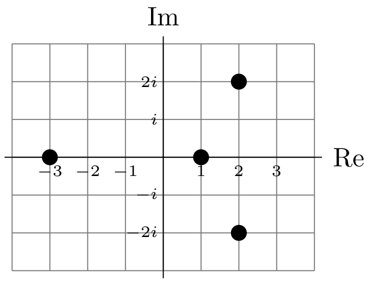
31. Find the value of a for which the remainder of $x^3 + (1-3a)x^2 + (2+a^2)x - 20$ divided by $x - 4$ is as small as possible.

A polynomial is called **irreducible** if it is *not* the product of two non-constant polynomials.

32. Factor the following polynomials into irreducible real factors:
(a) x^3+x^2+x+1 (b) x^3+x^2-x-1 (c) x^4-4x^3+8x (d) x^4+5x^2+6

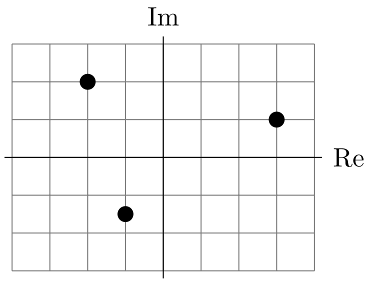
33. Factor the following polynomials into irreducible complex factors:
(a) z^3+z^2+z+1 (b) z^3+z^2-z-1 (c) z^4-4z^3+8z (d) z^4+5z^2+6

34. The figure below shows four points on the complex plane.



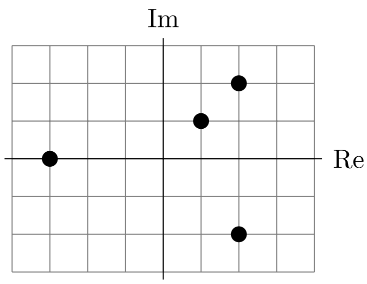
Give a monic polynomial of degree 4 whose roots are exactly these four points.

35. The figure below shows three points on the complex plane.



For which values of d does there exist a complex polynomial of degree d whose roots are exactly these three points?

36. The figure below shows four points on the complex plane.



(a) Does there exist a polynomial with real coefficients whose roots are exactly these four points? Why or why not?

(b) Does there exist a polynomial with complex coefficients whose roots are exactly these four points? Why or why not?

37. (a) For $f(x) = (z-3)^4(z+2)$, what is the multiplicity of 3?

    (b) For $f(x) = z^3 + 2z^2 - 7z + 4$, what is the multiplicity of 1?

38. The only roots of $z^5 - 4z^4 + z^3 + 10z^2 - 4z - 8$ are -1 (with some multiplicity) and $+2$ (with some multiplicity). What is the sum of these multiplicities?

39. The polynomial $f(z) = 3z^6 - 25z^5 + 62z^4 - 132z^3 + 89z^2 - 107z + 30$ has no repeated roots. How many linear polynomials are factors of $f(z)$?

40. (a) Write $\dfrac{5x+6}{x^2-6x+8}$ as a sum of partial fractions.

    (b) Write $\dfrac{36}{x^3+9x^2+18x}$ as a sum of partial fractions.

41. Given that $2x^3-13x^2+26x-10 = (2x-1)(x^2-6x+10)$, write $\dfrac{29}{2x^3-13x^2+26x-10}$ as a sum of partial fractions.