Linear Algebra, Winter 2021

List 4

Extra practice: complex numbers and polynomials.

42. (a) Simplify $x^9 \cdot x^4$. $x^{9+4} = \boxed{x^{13}}$

    (b) Simplify $e^9 \cdot e^4$. $e^{9+4} = \boxed{e^{13}}$

    (c) Simplify $5e^{90} \cdot 7e^{40}$. $(5 \cdot 7)e^{90+40} = \boxed{35e^{130}}$

    (d) Simplify $5e^{9i} \cdot 7e^{4i}$. $(5 \cdot 7)e^{9i+4i} = 35e^{(9+4)i} = \boxed{35e^{13i}}$

    (e) Simplify $5e^{(\pi/6)i} \cdot 7e^{(\pi/2)i}$. $(5 \cdot 7)e^{(\pi/6)i+(\pi/2)i} = \boxed{35e^{(2\pi/3)i}}$

43. [fast] With $z = 2\sqrt{3}e^{(\pi/3)i} = \sqrt{3} + 3i$ and $w = 2e^{(\pi/6)i} = \sqrt{3} + i$, compute $\dfrac{z}{w}$, giving your answer in either polar form or rectangular form (your choice).

    $$\frac{2\sqrt{3}e^{(\pi/3)i}}{2e^{(\pi/6)i}} = \frac{2\sqrt{3}}{2}e^{\left(\frac{\pi}{3}-\frac{\pi}{6}\right)i} = \boxed{\sqrt{3}e^{(\pi/6)i}} \text{ or } \boxed{\tfrac{3}{2} + \tfrac{\sqrt{3}}{2}i}$$

44. [fast] With $z = 2\sqrt{3}e^{(\pi/3)i} = \sqrt{3} + 3i$ and $w = 2e^{(\pi/6)i} = \sqrt{3} + i$, compute $z - w$, giving your answer in either polar form or rectangular form (your choice).

    $$\left(\sqrt{3} + 3i\right) - \left(\sqrt{3} + i\right) = \left(\sqrt{3} - \sqrt{3}\right) + (3i - i) = \boxed{2i} \text{ or } \boxed{2e^{(\pi/2)i}}$$

45. Convert the following numbers to polar form, that is, $\_\_\, e^{(\_\_ i)}$ where the first blank is a positive number.

    (a) $4\cos(21°) + 4\sin(21°)i$ $\boxed{4e^{21°\, i}}$

    (b) $9\cos(-3°) + 9\sin(-3°)i$ $\boxed{9e^{-3°\, i}}$ or $9e^{357°\, i}$

    (c) $\cos(\pi/4) + \sin(\pi/4)i$ $\boxed{e^{(\pi/4)i}}$

    (d) $\sqrt{2}\cos(\pi/4) + \sqrt{2}\sin(\pi/4)i$ $\boxed{\sqrt{2}e^{(\pi/4)i}}$

    (e) $1 + i$ $\boxed{\sqrt{2}e^{(\pi/4)i}}$ This is the same number as in (d).

    (f) $4i$ $\boxed{4e^{(\pi/2)i}}$

    (g) $\frac{1}{2} + \frac{\sqrt{3}}{2}i$ $\boxed{e^{(\pi/3)i}}$

    (h) $\frac{1}{2} - \frac{\sqrt{3}}{2}i$ $\boxed{e^{(-\pi/3)i}}$ or $e^{(5\pi/3)i}$

    (i) $1 - \sqrt{3}i$ $\boxed{2e^{(-\pi/3)i}}$ or $2e^{(5\pi/3)i}$

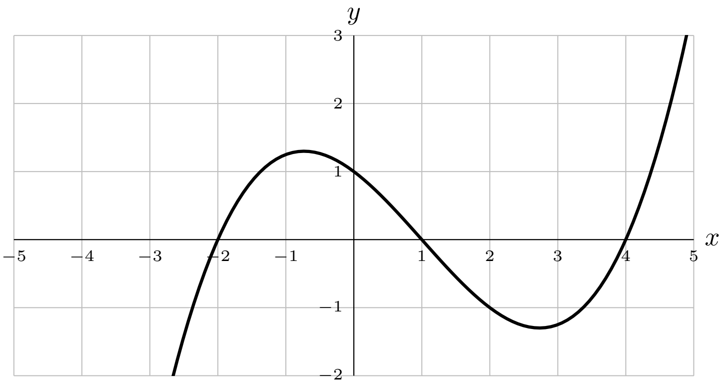
    (j) $\sqrt{3} - 3i$ $\boxed{2\sqrt{3}e^{(-\pi/3)i}}$ or $2\sqrt{3}e^{(5\pi/3)i}$

    (k) $\overline{-5 - 5i} = -5 + 5i = \boxed{5\sqrt{2}e^{(3\pi/4)i}}$

    (ℓ) $\overline{6e^{5\pi/6}}$ $\boxed{6e^{-5\pi/6}}$

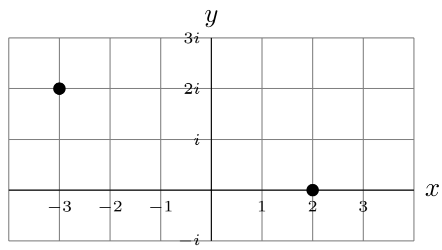
    (m) $\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)^{10} = \left(e^{\frac{\pi}{4}i}\right)^{10} = e^{\frac{5\pi}{2}i} = e^{\left(\frac{5\pi}{2}-2\pi\right)i} = \boxed{e^{\frac{\pi}{2}i}}$

46. Give the real polynomial of the form $\_\_\, x^3 + \_\_\, x^2 + \_\_\, x + \_\_$ whose graph is shown below. $\left(x - (-2)\right)\left(x - 1\right)\left(x - 4\right) = \boxed{x^3 + 4x^2 + x - 26}$

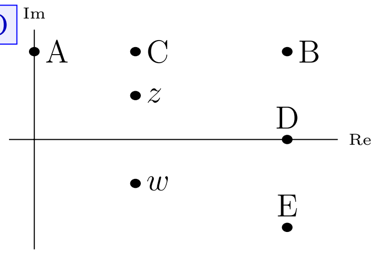


47. Give the real polynomial of the form $x^3 + \_\_ x^2 + \_\_ x + \_\_$ whose roots include the two complex numbers shown below.

$\big(x - (-3+2i)\big)\big(x - (-3-2i)\big)(x-2) = \boxed{x^3 + 4x^2 + x - 26}$



48. Which of the points A - E below could be $z \cdot w$? $\boxed{\text{D}}$



49. [fast] Find the real root(s) of $x^8 + x^6$ and the multiplicity of each root.

$\boxed{\text{0 has multiplicity 6.}}$

50. [medium fast] Find the complex roots of $x^8 + x^6$ and their multiplicities.

$\boxed{\text{0 has multiplicity 6. } i \text{ has multiplicity 1. } -i \text{ has multiplicity 1.}}$

51. [slow] One of the roots of $2x^5 - 5x^4 + 10x^2 - 10x + 3$ is 1; what is its multiplicity?

$f(x) = (x-1)^4(2x+3)$, so 1 has $\boxed{\text{multiplicity 4}}$

52. (a) [medium fast] Give the quotient when $x^3 + 2x^2 - 7x + 1$ is divided by $x - 2$.

$\boxed{x^2 + 4x + 1}$

(b) [fast] Give the remainder when $x^3 + 2x^2 - 7x + 1$ is divided by $x - 2$.

$R(x) = \text{constant}$. $R(2) = f(2) = \boxed{3}$

(c) [slow] Give the remainder when $x^3 + 2x^2 - 7x + 1$ is divided by $x^2 - 2$.

$\boxed{-5x + 5}$

53. [fast] The partial fraction decomposition of $\frac{x^5-23x^2+15x+81}{x^2-4x+4}$ is

$$\frac{x^5-23x^2+15x+81}{x^2-4x+4} = (x^3+4x^2+12x+9) + \frac{3}{x-2} + \frac{51}{(x-2)^2}.$$

Using this, give the quotient when $x^5-23x^2+15x+81$ is divided by x^2-4x+4.

Multiplying both sides of the equation by $(x-2)^2$ gives

$$x^5-23x^2+15x+81 = (x^3+4x^2+12x+9)(x^2-4x+4) + 3(x-2) + 51$$

or

$$x^5-23x^2+15x+81 = (x^3+4x^2+12x+9)(x^2-4x+4) + (3x+45),$$

so the remainder is $\boxed{3x+45}$.

54. [slow] Write $\dfrac{x^3+4x^2+x}{x^2+4x-5}$ as the sum of a polynomial and some partial fractions.

$$x^3+4x^2+x = Q\cdot(x^2+4x-5)+R \quad \leftarrow \text{quot. and rem.}$$
$$x^3+4x^2+x = (x)(x^2+4x-5)+(6x)$$
$$\frac{x^3+4x^2+x}{x^2+4x-5} = x + \frac{6x}{x^2+4x-5}$$

To split $\frac{6x}{x^2+4x-5}$ into partial fractions,

$$\frac{6x}{x^2+4x-5} = \frac{6x}{(x-1)(x+5)} = \frac{A}{x-1} + \frac{B}{x+5} \quad \text{for some } A, B$$
$$\frac{6x(x-1)(x+5)}{(x-1)(x+5)} = \frac{A(x-1)(x+5)}{x-1} + \frac{B(x-1)(x+5)}{x+5}$$
$$6x = A(x+5) + B(x-1)$$
$$6x = Ax + 5A + Bx - B$$
$$6x + 0 = (A+B)x + (5A-B), \text{ so } \left\{\begin{smallmatrix} A+B=6 \\ 5A-B=0 \end{smallmatrix}\right.$$

$A=1$ and $B=5$. Since we know $\frac{x^3+4x^2+x}{x^2+4x-5} = x + \frac{6x}{x^2+4x-5}$, we now have

$$\frac{x^3+4x^2+x}{x^2+4x-5} = \boxed{x + \frac{1}{x-1} + \frac{5}{x+5}}$$

55. [very slow] Give the partial fraction decomposition of $\dfrac{x^6+x^3+x^2+2}{x^3+x}$.

$$x^6+x^3+x^2+2 = (x^3-x+1)(x^3+x) + (2x^2-x+2)$$
$$\frac{x^6+x^3+x^2+2}{x^3+x} = (x^3-x+1) + \frac{2x^2-x+2}{x^3+x}$$
$$= (x^3-x+1) + \frac{2x^2-x+2}{x(x^2+1)}$$
$$= (x^3-x+1) + \frac{A}{x} + \frac{Bx+C}{x^2+1} \quad \text{for some } A, B, C$$
$$= \boxed{(x^3-x+1) + \frac{2}{x} + \frac{-1}{x^2+1}}$$