Linear Algebra, Winter 2021

List 5

Vectors

56. For each expression below, does it represent a scalar, a vector, or nonsense? (By "nonsense" we mean it is not a legal operation; for example, $\vec{v}+5$ is nonsense.)

(a) $\vec{a}+\vec{b}$ vector

(b) $\vec{u}\cdot\vec{v}$ scalar

(c) $\vec{a}\vec{b}$ nonsense

(d) $t\vec{a}$ vector

(e) $t+\vec{v}$ nonsense

(f) $(t+s)\vec{u}$ vector

(g) $\vec{n}/s$ vector

(h) $\vec{a}-s$ nonsense

(i) $\vec{c}+s\vec{b}$ vector

(j) $t(\vec{a}+\vec{b})-\vec{c}$ vector

(k) $(\vec{a}\cdot\vec{b})\vec{c}$ vector

(l) $\vec{0}-\vec{a}$ vector

(m) $\vec{0}\cdot\vec{w}$ scalar

(n) $\begin{bmatrix}4\\2\end{bmatrix}\cdot\begin{bmatrix}1\\8\end{bmatrix}$ scalar

(o) $\langle 4,2\rangle\cdot\langle s,t\rangle$ scalar

(p) $\vec{w}\cdot\langle s,t\rangle$ scalar

(q) $|\vec{u}|$ scalar

(r) $|\langle 9,2,\frac{1}{2}\rangle|$ scalar

(s) $|\vec{w}|\vec{v}$ vector

(t) $|\vec{a}|+(\vec{b}\cdot\vec{c})$ scalar

(u) $|\vec{a}|(\vec{b}\cdot\vec{c})$ scalar

(v) $(\vec{a})^2$ nonsense

(w) $|\vec{a}|^2$ scalar

In 2D, the **zero vector** is $\vec{0}=\langle 0,0\rangle$, and the **standard basis vectors** are $\vec{\imath}=\langle 1,0\rangle$ and $\vec{\jmath}=\langle 0,1\rangle$.
In 3D, the **zero vector** is $\vec{0}=\langle 0,0,0\rangle$, and the **standard basis vectors** are $\vec{\imath}=\langle 1,0,0\rangle$ and $\vec{\jmath}=\langle 0,1,0\rangle$ and $\vec{k}=\langle 0,0,1\rangle$.

In any dimension, the **magnitude** (or **length**) of a vector is

$$|\langle v_1,\ldots,v_n\rangle| = \sqrt{(v_1)^2+(v_2)^2+\cdots+(v_n)^2},$$

the **scalar multiplication** of $\vec{v}$ by s is

$$s\langle v_1,\ldots,v_n\rangle = \langle sv_1,\ldots,sv_n\rangle,$$

and **vector addition** of $\vec{u}$ and $\vec{v}$ is

$$\langle u_1,\ldots,u_n\rangle+\langle v_1,\ldots,v_n\rangle = \langle u_1+v_1,\ldots,u_n+v_n\rangle.$$

57. Calculate each of the following:

(a) $\langle 3,2\rangle+\langle 7,1\rangle$ $\langle 10,3\rangle$ or $\begin{bmatrix}10\\3\end{bmatrix}$ or $10\hat{\imath}+3\hat{\jmath}$, etc.

(b) $\begin{bmatrix}3\\2\end{bmatrix}-\begin{bmatrix}7\\1\end{bmatrix}$ $\langle -4,1\rangle$ (or other formats)

(c) $\langle 3,2\rangle\cdot\langle 7,1\rangle = 3(7)+2(1)=21+2=23$

(d) $8\begin{bmatrix}3\\2\end{bmatrix}+\frac{1}{2}\begin{bmatrix}7\\1\end{bmatrix} = \begin{bmatrix}24\\16\end{bmatrix}+\begin{bmatrix}3.5\\0.5\end{bmatrix} = \begin{bmatrix}27.5\\16.5\end{bmatrix}$ or $\begin{bmatrix}55/2\\33/2\end{bmatrix}$

(e) $|\langle 3,2\rangle|$ $\sqrt{3^2+2^2}=\sqrt{9+4}=\sqrt{13}$

(f) $\frac{1}{20}\langle 3,2\rangle$ $\langle\frac{3}{20},\frac{1}{10}\rangle$

(g) $\dfrac{\langle 3,2\rangle}{|\langle 3,2\rangle|}$ $\langle\frac{3}{\sqrt{13}},\frac{2}{\sqrt{13}}\rangle$

(h) $9\langle 1,0\rangle + 2\langle 0,1\rangle$ $\boxed{\langle 9,2\rangle}$

(i) $9\vec{\imath} + 2\vec{\jmath}$ (in 2D) $\boxed{\langle 9,2\rangle}$

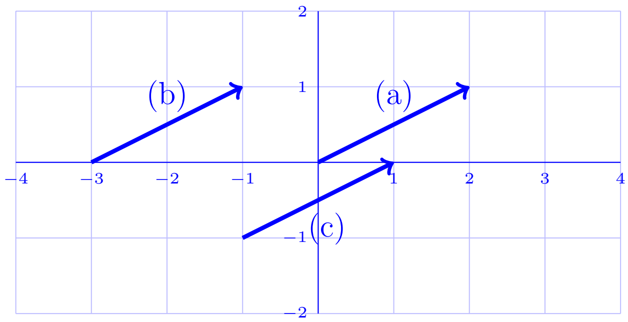
(j) $6\vec{\imath} + \vec{\jmath} - 2\vec{k}$ $\boxed{\langle 6,1,-2\rangle}$

58. Draw the following vectors as arrows all on the same plane (one drawing, not three drawings):

(a) the vector $2\vec{\imath} + \vec{\jmath}$ with its tail at $(0,0)$.

(b) the vector $2\vec{\imath} + \vec{\jmath}$ with its tail at $(-3,0)$.

(c) the vector $2\vec{\imath} + \vec{\jmath}$ with its tail at $(-1,-1)$.



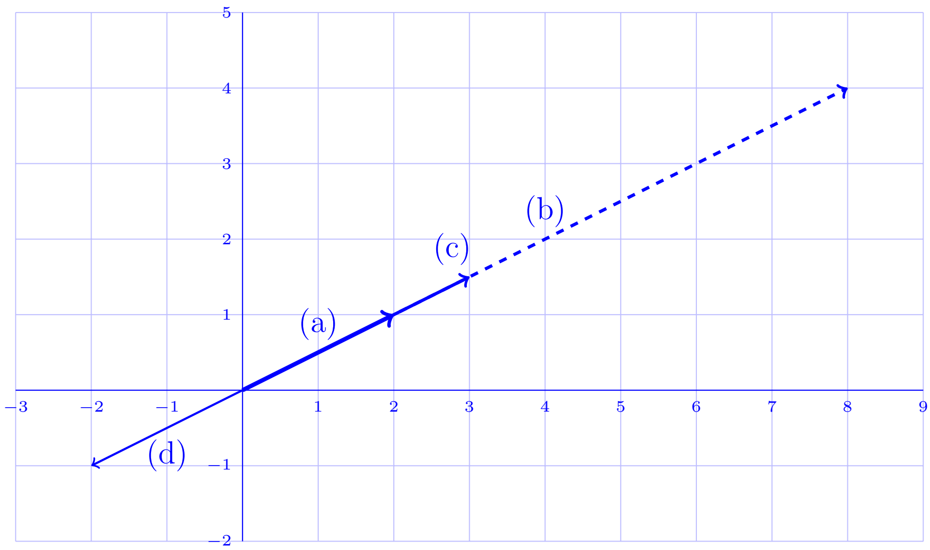
59. Draw the following vectors as arrows all on the same plane (one drawing, not four drawings):

(a) the vector $\langle 2,1\rangle$ with its tail at $(0,0)$.

(b) the vector $4\langle 2,1\rangle$ with its tail at $(0,0)$.

(c) the vector $1.5\langle 2,1\rangle$ with its tail at $(0,0)$.

(d) the vector $(-1)\langle 2,1\rangle$ with its tail at $(0,0)$.



60. Let P be the point $(5, 2)$ and let Q be the point $(1, 9)$. Describe the vector $\left[\begin{smallmatrix}5\\2\end{smallmatrix}\right] - \left[\begin{smallmatrix}1\\9\end{smallmatrix}\right]$ in words, *without doing any calculations.*

$\boxed{\text{An arrow from } Q \text{ to } P}$

61. Calculate each of the following. Each answer will be either a scalar *expression* or a vector *expression* involving t.

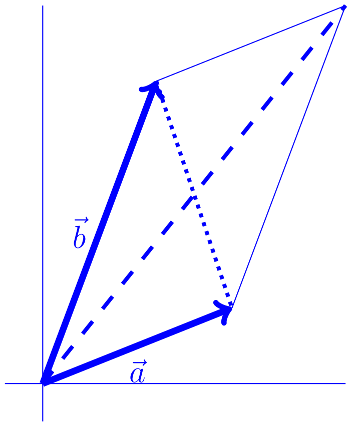
(a) $5\begin{bmatrix}3\\2\end{bmatrix} + t\begin{bmatrix}7\\1\end{bmatrix}$ $\boxed{\begin{bmatrix}15+7t\\10+t\end{bmatrix}}$

(b) $t + \left|\left[\begin{smallmatrix}3\\2\end{smallmatrix}\right]\right|$ $\boxed{t+\sqrt{13}}$

(c) $|t\langle 3, 2\rangle|$ $\boxed{t\sqrt{13}}$

(d) $|\langle 1+t, 1-t\rangle|^2 = \left(\sqrt{(1+t)^2+(1-t)^2}\right)^2 = \boxed{(1+t)^2+(1-t)^2}$ or $\boxed{2+2t^2}$

62. A parallelogram has the vector $\vec{a} = \begin{bmatrix}5\\2\end{bmatrix}$ along one edge and $\vec{b} = \begin{bmatrix}3\\8\end{bmatrix}$ along another edge. Compute the lengths of the two diagonals of the parallelogram.



The longer diagonal (dashed) is $\vec{a}+\vec{b}$. Its length is

$$|\vec{a}+\vec{b}| = |\langle 5+3, 2+8\rangle| = |\langle 8, 10\rangle|$$
$$= \sqrt{8^2+10^2} = \boxed{\sqrt{164}}$$

The shorter diagonal (dotted) is $\vec{a}-\vec{b}$. Its length is

$$|\vec{a}-\vec{b}| = |\langle 5-3, 2-8\rangle| = |\langle 2, -6\rangle|$$
$$= \sqrt{2^2+(-6)^2} = \boxed{\sqrt{40}}$$

63. Which of the following are scalar multiples of $\langle 4, 2, -6\rangle$?

(a) $\begin{bmatrix}20\\10\\-60\end{bmatrix}$ $\boxed{\text{No}}$

(b) $\langle -12, -6, 18\rangle$ Yes

(c) $\langle 0, 0, 0\rangle$ Yes

(d) $\begin{bmatrix}0.4\\0.2\\-0.6\end{bmatrix}$ Yes

(e) $\begin{bmatrix}\sqrt{32}\\\sqrt{8}\\-\sqrt{72}\end{bmatrix}$ Yes

(f) $\langle 8, 4, -10\rangle$ No

$\boxed{\text{(b), (c), (d), (e)}}$

64. Find the unit vector in the same direction as $\vec{v} = \langle 8, -1, 4\rangle$.

$$\frac{\langle 8, -1, 4\rangle}{|\langle 8, -1, 4\rangle|} = \frac{\langle 8, 1, 4\rangle}{\sqrt{81}} = \boxed{\left\langle \tfrac{8}{9}, \tfrac{-1}{9}, \tfrac{4}{9}\right\rangle}$$

The **dot product** (also called **scalar product**) of $\vec{u}$ and $\vec{v}$ is written $\vec{u}\cdot\vec{v}$ and can be calculated as either

$$\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n$$

or

$$\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos(\text{angle between } \vec{u} \text{ and } \vec{v}).$$

Two vectors are called **orthogonal** if their dot product is 0.

65. Give the two vectors that are orthogonal to $\vec{v} = \left[\begin{smallmatrix}5\\12\end{smallmatrix}\right]$ and have length 1.

Any scalar multiple of $\langle 5, -12\rangle$ is orthogonal to $\langle 5, 12\rangle$. To have length 1 we can use only

$$\frac{\langle 5,-12\rangle}{|\langle 5,12\rangle|} = \frac{\langle 5,-12\rangle}{\sqrt{25+144}} = \frac{\langle 5,-12\rangle}{13} = \boxed{\left\langle \tfrac{5}{13}, \tfrac{-12}{13}\right\rangle}$$

or $\boxed{\left\langle \tfrac{-5}{13}, \tfrac{12}{13}\right\rangle}$.

66. Give an example of a vector that is perpendicular to $\vec{v} = \left[\begin{smallmatrix}1\\9\\4\end{smallmatrix}\right]$.

There are many, many correct answers. One correct answer is $\langle 4, 0, -1\rangle$.

67. Write $\left[\begin{smallmatrix}A\\B\\C\end{smallmatrix}\right]\cdot\left(\left[\begin{smallmatrix}x\\y\\z\end{smallmatrix}\right]-\left[\begin{smallmatrix}a\\b\\c\end{smallmatrix}\right]\right)$ as an expression that does not use any vector notation.

$\boxed{A(x-a)+B(y-b)+C(z-c)}$

68. Knowing that

$$\cos(19.5^\circ)\approx\tfrac{33}{35},\quad \cos(25.2^\circ)\approx\tfrac{19}{21},\quad \cos(31^\circ)\approx\tfrac{6}{7},\quad \cos(62.96^\circ)\approx\tfrac{15}{33},$$

find the acute angle between $\langle 6,3,6\rangle$ and $\langle 6,9,18\rangle$.

We know

$$\langle 6,3,6\rangle\cdot\langle 6,9,18\rangle = 6(6)+3(9)+6(18) = 171$$

and also

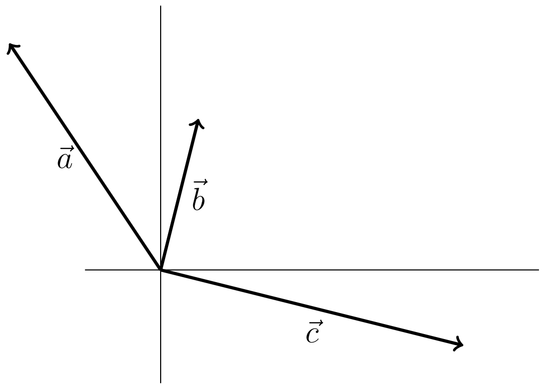
$$\langle 6,3,6\rangle\cdot\langle 6,9,18\rangle = |\langle 6,3,6\rangle|\,|\langle 6,9,18\rangle|\,\cos\theta = (9)(21)\cos\theta = 189\cos\theta,$$

so it must be that

$$171 = 189\cos\theta \qquad\Rightarrow\qquad \cos\theta = \frac{171}{189} = \frac{19}{21}$$

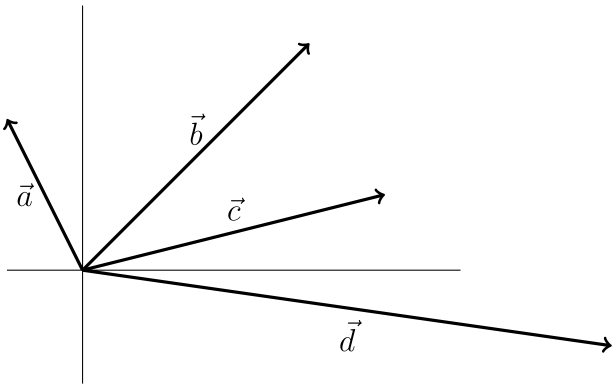
and therefore $\boxed{\theta = 25.2^\circ}$.

69. Let $\vec{a}$, $\vec{b}$, and $\vec{c}$ be as in the image below. Which of $\vec{a}\cdot\vec{b}$, $\vec{a}\cdot\vec{c}$, and $\vec{b}\cdot\vec{c}$ is zero?



$\boxed{\vec{b}\cdot\vec{c}}$ because they make a $90°$ angle, and $\cos(90°) = 0$.

70. Let $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ be as in the image below.



(a) Write two true equations of the form __ + __ = __ using these vectors.
$\vec{a}+\vec{c}=\vec{b}$ and $\vec{c}+\vec{a}=\vec{b}$.

(b) Write two true equations of the form __ − __ = __ using these vectors.
$\vec{b}-\vec{a}=\vec{c}$ and $\vec{b}+\vec{c}=\vec{a}$.