Linear Algebra, Winter 2021

List 5

Vectors

56. For each expression below, does it represent a scalar, a vector, or nonsense? (By "nonsense" we mean it is not a legal operation; for example, $\vec{v} + 5$ is nonsense.)

(a) $\vec{a} + \vec{b}$
(b) $\vec{u} \cdot \vec{v}$
(c) $\vec{a}\vec{b}$
(d) $t\vec{a}$
(e) $t + \vec{v}$
(f) $(t+s)\vec{u}$
(g) $\vec{n}/s$
(h) $\vec{a} - s$
(i) $\vec{c} + s\vec{b}$
(j) $t(\vec{a} + \vec{b}) - \vec{c}$
(k) $(\vec{a} \cdot \vec{b})\vec{c}$
(l) $\vec{0} - \vec{a}$
(m) $\vec{0} \cdot \vec{w}$
(n) $\begin{bmatrix} 4 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 8 \end{bmatrix}$
(o) $\langle 4, 2 \rangle \cdot \langle s, t \rangle$
(p) $\vec{w} \cdot \langle s, t \rangle$
(q) $|\vec{u}|$
(r) $\left|\langle 9, 2, \frac{1}{2} \rangle\right|$
(s) $|\vec{w}|\vec{v}$
(t) $|\vec{a}| + (\vec{b} \cdot \vec{c})$
(u) $|\vec{a}|(\vec{b} \cdot \vec{c})$
(v) $(\vec{a})^2$
(w) $|\vec{a}|^2$

In 2D, the **zero vector** is $\vec{0} = \langle 0, 0 \rangle$, and the **standard basis vectors** are $\vec{\imath} = \langle 1, 0 \rangle$ and $\vec{\jmath} = \langle 0, 1 \rangle$.
In 3D, the **zero vector** is $\vec{0} = \langle 0, 0, 0 \rangle$, and the **standard basis vectors** are $\vec{\imath} = \langle 1, 0, 0 \rangle$ and $\vec{\jmath} = \langle 0, 1, 0 \rangle$ and $\vec{k} = \langle 0, 0, 1 \rangle$.

In any dimension, the **magnitude** (or **length**) of a vector is

$$\left|\langle v_1, ..., v_n \rangle\right| = \sqrt{(v_1)^2 + (v_2)^2 + \cdots + (v_n)^2},$$

the **scalar multiplication** of $\vec{v}$ by s is

$$s\langle v_1, ..., v_n \rangle = \langle sv_1, ..., sv_n \rangle,$$

and **vector addition** of $\vec{u}$ and $\vec{v}$ is

$$\langle u_1, ..., u_n \rangle + \langle v_1, ..., v_n \rangle = \langle u_1 + v_1, ..., u_n + v_n \rangle.$$

57. Calculate each of the following:

(a) $\langle 3, 2 \rangle + \langle 7, 1 \rangle$
(b) $\begin{bmatrix} 3 \\ 2 \end{bmatrix} - \begin{bmatrix} 7 \\ 1 \end{bmatrix}$
(c) $\langle 3, 2 \rangle \cdot \langle 7, 1 \rangle$
(d) $8\begin{bmatrix} 3 \\ 2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 7 \\ 1 \end{bmatrix}$
(e) $|\langle 3, 2 \rangle|$
(f) $\frac{1}{20}\langle 3, 2 \rangle$
(g) $\dfrac{\langle 3, 2 \rangle}{|\langle 3, 2 \rangle|}$
(h) $9\langle 1, 0 \rangle + 2\langle 0, 1 \rangle$
(i) $9\vec{\imath} + 2\vec{\jmath}$ (in 2D)
(j) $6\vec{\imath} + \vec{\jmath} - 2\vec{k}$

58. Draw the following vectors as arrows all on the same plane (one drawing, not three drawings):

(a) the vector $2\vec{\imath} + \vec{\jmath}$ with its tail at $(0, 0)$.
(b) the vector $2\vec{\imath} + \vec{\jmath}$ with its tail at $(-3, 0)$.
(c) the vector $2\vec{\imath} + \vec{\jmath}$ with its tail at $(-1, -1)$.

59. Draw the following vectors as arrows all on the same plane (one drawing, not four drawings):

    (a) the vector $\langle 2,1\rangle$ with its tail at $(0,0)$.

    (b) the vector $4\langle 2,1\rangle$ with its tail at $(0,0)$.

    (c) the vector $1.5\langle 2,1\rangle$ with its tail at $(0,0)$.

    (d) the vector $(-1)\langle 2,1\rangle$ with its tail at $(0,0)$.

60. Let P be the point $(5,2)$ and let Q be the point $(1,9)$. Describe the vector $\left[\begin{smallmatrix}5\\2\end{smallmatrix}\right]-\left[\begin{smallmatrix}1\\9\end{smallmatrix}\right]$ in words, *without doing any calculations.*

61. Calculate each of the following. Each answer will be either a scalar *expression* or a vector *expression* involving t.

    (a) $5\begin{bmatrix}3\\2\end{bmatrix}+t\begin{bmatrix}7\\1\end{bmatrix}$

    (b) $t+\left|\left[\begin{smallmatrix}3\\2\end{smallmatrix}\right]\right|$

    (c) $\left|t\langle 3,2\rangle\right|$

    (d) $\left|\langle 1+t,1-t\rangle\right|^2$

62. A parallelogram has the vector $\vec{a}=\begin{bmatrix}5\\2\end{bmatrix}$ along one edge and $\vec{b}=\begin{bmatrix}3\\8\end{bmatrix}$ along another edge. Compute the lengths of the two diagonals of the parallelogram.

63. Which of the following are scalar multiples of $\langle 4,2,-6\rangle$?

    (a) $\begin{bmatrix}20\\10\\-60\end{bmatrix}$

    (b) $\langle -12,-6,18\rangle$

    (c) $\langle 0,0,0\rangle$

    (d) $\begin{bmatrix}0.4\\0.2\\-0.6\end{bmatrix}$

    (e) $\begin{bmatrix}\sqrt{32}\\\sqrt{8}\\-\sqrt{72}\end{bmatrix}$

    (f) $\langle 8,4,-10\rangle$

64. Find the unit vector in the same direction as $\vec{v}=\langle 8,-1,4\rangle$.

The **dot product** (also called **scalar product**) of $\vec{u}$ and $\vec{v}$ is written $\vec{u}\cdot\vec{v}$ and can be calculated as either

$$\vec{u}\cdot\vec{v}=u_1v_1+u_2v_2+\cdots+u_nv_n$$

or

$$\vec{u}\cdot\vec{v}=|\vec{u}||\vec{v}|\cos(\text{angle between }\vec{u}\text{ and }\vec{v}).$$

Two vectors are called **orthogonal** if their dot product is 0.

65. Give the two vectors that are orthogonal to $\vec{v}=\left[\begin{smallmatrix}5\\12\end{smallmatrix}\right]$ and have length 1.

66. Give an example of a vector that is perpendicular to $\vec{v}=\left[\begin{smallmatrix}1\\9\\4\end{smallmatrix}\right]$.

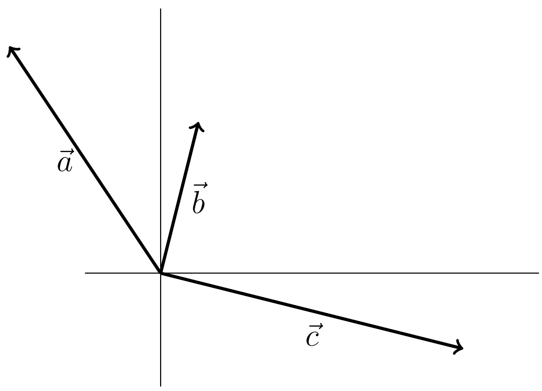
67. Write $\left[\begin{smallmatrix}A\\B\\C\end{smallmatrix}\right]\cdot\left(\left[\begin{smallmatrix}x\\y\\z\end{smallmatrix}\right]-\left[\begin{smallmatrix}a\\b\\c\end{smallmatrix}\right]\right)$ as an expression that does not use any vector notation.

68. Knowing that

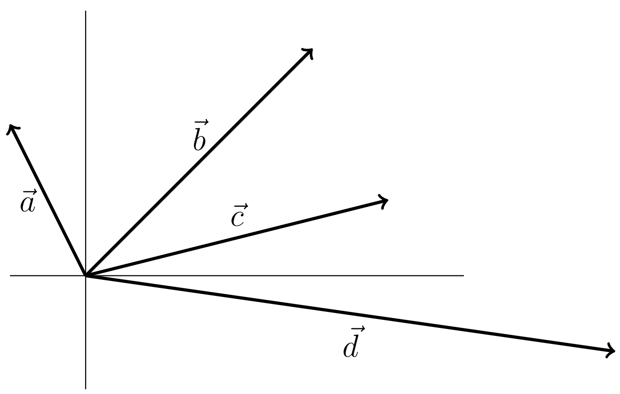
$$\cos(19.5°) \approx \tfrac{33}{35}, \quad \cos(25.2°) \approx \tfrac{19}{21}, \quad \cos(31°) \approx \tfrac{6}{7}, \quad \cos(62.96°) \approx \tfrac{15}{33},$$

find the acute angle between $\langle 6,3,6\rangle$ and $\langle 6,9,18\rangle$.

69. Let $\vec{a}$, $\vec{b}$, and $\vec{c}$ be as in the image below. Which of $\vec{a}\cdot\vec{b}$, $\vec{a}\cdot\vec{c}$, and $\vec{b}\cdot\vec{c}$ is zero?



70. Let $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ be as in the image below.



(a) Write two true equations of the form __ + __ = __ using these vectors.

(b) Write two true equations of the form __ − __ = __ using these vectors.