Linear Algebra, Winter 2021

List 6

Lines and planes, cross product

The vector $\vec{r}$ is used for $\langle x, y\rangle$ in 2D and $\langle x, y, z\rangle$ in 3D.

The line through point $\vec{p}$ parallel to vector $\vec{d}$ has parametric vector equation

$$\vec{r} = \vec{p} + t\vec{d}.$$

The vector $\vec{d}$ is called a **direction vector** for the line. This vector equation is a short way of writing the 2 or 3 parametric equations with only scalars.

The plane through point $\vec{p}$ parallel to vectors $\vec{u}$ and $\vec{v}$ has parametric vector eqn.

$$\vec{r} = \vec{p} + t\vec{u} + s\vec{v}.$$

The plane through $\vec{p}$ perpendicular to vector $\vec{n}$ has vector equation

$$\vec{n} \cdot (\vec{r} - \vec{p}) = 0.$$

The vector $\vec{n}$ is called a **normal vector** for the plane. If $\vec{p} = \langle x_0, y_0, z_0\rangle$ and $\vec{n} = \langle a, b, c\rangle$, the vector equation above is equivalent to the "scalar equation"

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

or to

$$ax + by + cz = D$$

for the number $D = \vec{p} \cdot \vec{n}$.

71. Write an equation in vector form for the line through points $(2, 3)$ and $(-3, 7)$.

    The vector from P_1 to P_2 is one possible direction vector for this line. That vector is

    $$\vec{v} = \vec{P_2} - \vec{P_1} = \langle -3, 7\rangle - \langle 2, 3\rangle = \langle -5, 4\rangle.$$

    One equation for the line is $\vec{r} = \vec{P_1} + t\vec{v}$, which is

    $$\boxed{\vec{r} = \langle 2, 3\rangle + t\langle -5, 4\rangle}.$$

72. Describe the line through the point $(2,\ 2.4,\ 3.5)$ and parallel to the vector $3\,\vec{\imath} + 2\,\vec{\jmath} - \vec{k}$ using the parameter t in

    (a) one vector equation. $\boxed{\vec{r} = \langle 2, 2.4, 3.5\rangle + t\langle 3, 2, -1\rangle}$

    (b) three scalar equations. $\boxed{x = 2 + 3t, \quad y = 2.4 + 2t, \quad z = 3.5 - t}$

73. Does the point $(4, 8, 7)$ line on the line $\vec{r} = \langle 1, 2, 6\rangle + \langle 3, 8, 9\rangle t$?

74. Find a scalar equation for the plane through the origin perpendicular to the vector $\langle 1, -2, 5\rangle$.

    Since $\langle 1, -2, 5\rangle \cdot \langle x, y, z\rangle = 0$, this is $\boxed{x - 2y + 5 = 0}$.

75. Find an equation for the plane through the point $(1, -1, -1)$ parallel to the plane $5x - y - z = 6$.

The first plane is normal to $\vec{n} = \langle 5, -1, -1\rangle$, so this one is too. One equation is

$$\boxed{\langle 5, -1, -1\rangle \cdot (\langle x, y, z\rangle - \langle 1, -1, -1\rangle) = 0.}$$

By expanding the dot product, we get the scalar equation

$$\boxed{5(x-1) + (-1)(y+1) + (-1)(z+1) = 0,}$$

which can be simplified to just $\boxed{5x - y - z = 7}$.

76. Find the distance from the point $(-6, 3, 5)$ to the plane $x - 2y - 4z = 8$.

The closest point to $(-6, 3, 5)$ on the plane $x - 2y - 4z = 8$ is on the line through $(-6, 3, 5)$ that is perpendicular to $x - 2y - 4z = 8$. That line has direction vector $\vec{v} = \langle 1, -2, -4\rangle$, the same as the normal vector for the plane, so that line is

$$\vec{r} = \langle -6, 3, 5\rangle + \langle 1, -2, -4\rangle t$$

or

$$\vec{r} = \langle -6 + t, 3 - 2t, 5 - 4t\rangle.$$

The intersection of this line with $x - 2y - 4z = 8$ occurs when

$$\begin{aligned}(-6+t) - 2(3-2t) - 4(5-4t) &= 8\\ -32 + 21t &= 8\\ t &= 41/21\end{aligned}$$

which corresponds to the point

$$\langle -6 + \tfrac{41}{21}, 3 - 2\tfrac{41}{21}, 5 - 4\tfrac{41}{21}\rangle = \langle \tfrac{-86}{21}, \tfrac{-17}{21}, \tfrac{-55}{21}\rangle.$$

The distance between $(-6, 3, 5)$ and $(\frac{-86}{21}, \frac{-17}{21}, \frac{-55}{21})$ is

$$\sqrt{(-6 - \tfrac{-86}{21})^2 + (3 - \tfrac{-17}{21})^2 + (5 - \tfrac{-55}{21})^2} = \boxed{\frac{40}{\sqrt{21}}}.$$

77. Explain why the parametric equations

$$\begin{cases} x = 1 - t \\ y = 2 - 3t \\ z = 4t \end{cases} \quad \text{and} \quad \begin{cases} x = 2s \\ y = -1 + 6s \\ z = 4 - 8s \end{cases}$$

describe the same line.

The direction vectors $\vec{v_1} = \langle -1, -3, 4\rangle$ and $\vec{v_2} = \langle 2, 6, -8\rangle$ are parallel, and the point $(0, -1, 4)$ is on both lines (this point corresponds to $t = 1$ for the first line and to $s = 0$ for the second line).

You could use any point that is on both lines in this explanation; you do not have to use $(0, -1, 4)$.

78. Find the intersection point of the line

$$L: \qquad x = t,\ y = 1 - 2t,\ z = -3 + 2t$$

and the plane

$$P: \qquad 3x - y - 2z = 3.$$

The value of t for this intersection point satisfies

$$\begin{aligned} 3x - y - 2z &= 3 \\ 3(t) - (1 - 2t) - 2(-3 + 2t) &= 3 \\ t + 5 &= 3 \\ t &= -2 \end{aligned}$$

The coordinates of this point are therefore

$$x = -2, \quad y = 1 - 2(-2) = 5, \quad z = -3 + 2(-2) = -7,$$

so the point is $\boxed{(-2, 5, -7)}$. (A previous file incorrectly wrote $(0, 1, -3)$ as the final answer.)

79. Do the lines

$$\begin{aligned} L_1: &\qquad x = 3 + t,\ y = 2 - 2t,\ z = 1 + 5t \\ L_2: &\qquad x = 8 - 3s,\ y = 1 + s,\ z = -10 + 5s \end{aligned}$$

intersect? If so, find the intersection point and the angle between the two lines at that point (using a calculator for $\cos^{-1}$).

The lines intersect at the point $\boxed{(13/5,\ 14/5, -1)}$ (this corresponds to $t = -2/5$ for L_1 and $s = 9/5$ for L_2). The angle between the lines is the angle between the direction vectors $\vec{a} = \langle 1, -2, 5\rangle$ and $\vec{b} = \langle -3, 1, 5\rangle$, which is

$$\arccos\left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}\right) = \boxed{\arccos\left(\frac{20}{5\sqrt{42}}\right) \approx 0.9056 \approx 51.89^\circ}.$$

80. Do the lines

$$\begin{aligned} L_1: &\qquad x = 3 + t,\ y = 2 - 2t,\ z = 1 + 5t \\ L_3: &\qquad x = 5 + 2s,\ y = -6 - s,\ z = 7 - 4s \end{aligned}$$

intersect? If so, find the intersection point and the angle between the two lines at that point (using a calculator for $\cos^{-1}$).

No intersection

81. Give parametric equations (with two parameters) for the plane through the origin parallel to both $\langle 1, 5, 9\rangle$ and $\langle 2, -1, 2\rangle$.

$\boxed{\vec{r} = t\langle 1, 5, 9\rangle + s\langle 2, -1, 2\rangle}$

The **cross product** of $\vec{a} = \langle a_1, a_2, a_3\rangle$ and $\vec{b} = \langle b_1, b_2, b_3\rangle$ is written $\vec{a} \times \vec{b}$ and can be calculated as

$$\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2)\vec{\imath} + (a_3b_1 - a_1b_3)\vec{\jmath} + (a_1b_2 - a_2b_1)\vec{k}.$$

This vector is perpendicular to $\vec{a}$ and to $\vec{b}$, has length $|\vec{a}||\vec{b}|\sin(\theta)$ where θ is the angle between $\vec{a}$ and $\vec{b}$, and its direction obeys the Right-Hand Rule.

82. For each expression below, does it represent a scalar, a vector, or nonsense?

(a) $\vec{a} \times t$ nonsense

(b) $\vec{a} \times \vec{b}$ vector

(c) $(\vec{a} \times b) \times \vec{c}$ vector

(d) $(\vec{u} + \vec{w}) \times \vec{v}$ vector

(e) $(\vec{u} \times \vec{w}) + \vec{v}$ vector

(f) $t(\vec{u} \times \vec{w}) + s\vec{v}$ vector

(g) $a\vec{u} \times (b\vec{v} + c\vec{w})$ vector

(h) $\begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix} \times \vec{w}$ vector

(i) $\langle 1, 3, 8 \rangle \times (\vec{a} + \vec{b})$ v.

(j) $\langle t, 4, s \rangle \times (\vec{a} - \vec{b})$ v.

(k) $|\vec{v} \times \vec{w}|$ scalar

(l) $5\vec{a} + \left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \times \vec{a} \right)$ v.

(m) $||\vec{v} \times \vec{w}||$ scalar

(n) $\vec{0} \times \vec{0}$ vector

(o) $(\vec{u} \times \vec{v}) \cdot \vec{w}$ scalar

(p) $\vec{u} \times (\vec{v} \cdot \vec{w})$ non.

(q) $|\langle 7, 4, 2 \rangle| \times \vec{v}$ non.

(r) $\vec{a} \times \vec{b} + t$ nonsense

(s) $|\vec{v} \times \vec{w}| - |\vec{v} \cdot \vec{w}|$ scalar

83. For the vectors $\vec{v} = \langle 2, -1, 3 \rangle$ and $\vec{w} = \langle 4, 2, 1 \rangle$,

(a) Give the vector $\vec{v} \times \vec{w}$. $\boxed{\langle -7, 10, 8 \rangle}$

(b) Give the vector $\vec{w} \times \vec{v}$. $\boxed{\langle 7, -10, -8 \rangle}$ Note that $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$.

(c) Give the scalar $\vec{v} \cdot (\vec{v} \times \vec{w})$. $\boxed{0}$ You could calculate $2(-7) + (-1)10 + 3(8) = 0$, but in fact $\vec{v} \cdot (\vec{v} \times \vec{w})$ will be 0 for ANY vectors because $\vec{v} \times w$ is always perpendicular to $\vec{v}$.

(d) Give the scalar $\vec{w} \cdot (\vec{v} \times \vec{w})$. $\boxed{0}$

84. Find parametric equations for the line through $(2, 1, 0)$ perpendicular to both $\vec{\imath} + \vec{\jmath}$ and $\vec{\jmath} + \vec{k}$.

This line is parallel to $(\vec{\imath} + \vec{\jmath}) \times (\vec{\jmath} + \vec{k}) = \langle 1, -1, 1 \rangle$. As an alternative to using cross-product, you can find any 3 numbers a, b, c for which

$$\langle 1, 1, 0 \rangle \cdot \langle a, b, c \rangle = 0 \qquad a + b = 0$$
$$\langle 0, 1, 1 \rangle \cdot \langle a, b, c \rangle = 0 \qquad b + c = 0$$

and use $\vec{d} = \langle a, b, c \rangle$. (All these $\langle a, b, c \rangle$ will be scalar multiples of $\langle 1, -1, 1 \rangle$.) Parametric equations for the line through $(2, 1, 0)$ parallel to $\langle 1, -1, 1 \rangle$ are $\boxed{x = 2 + t, \quad y = 1 - t, \quad z = t}$.

85. Give a scalar equation (with no parameters) for the plane from #81.

This plane is perpendicular to $\langle 1, 5, 9 \rangle \times \langle 2, -1, 2 \rangle = \langle 19, 16, -11 \rangle$, so its equation is $\boxed{19x + 16y - 11z = 0}$.

86. If $\vec{a} \cdot \vec{b} = \sqrt{3}$ and $\vec{a} \times \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, find the angle between $\vec{a}$ and $\vec{b}$.

The magnitude of $\langle 1, 2, 2 \rangle$ is $\sqrt{1 + 4 + 4} = \sqrt{9} = 3$. This gives two equations involving the angle θ between $\vec{a}$ and $\vec{b}$:

$$|\vec{a} \times \vec{b}| = |\vec{a}| \cdot |\vec{b}| \cdot \sin\theta = 3$$
$$\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos\theta = \sqrt{3}$$

Dividing the first equation by the second gives $\frac{\sin\theta}{\cos\theta} = \frac{3}{\sqrt{3}}$, or $\tan\theta = \sqrt{3}$. Thus $\theta = \tan^{-1}(\sqrt{3}) = \pi/3$ or $60°$.

87. If $|\vec{v}| = 3$, $|\vec{w}| = 5$, and $\vec{v} \cdot \vec{w} = 10$, compute $(\vec{v} + \vec{w}) \cdot (\vec{v} + 2\vec{w})$.

Using the fact that $\vec{v} \cdot \vec{v} = |\vec{v}||\vec{v}|\cos(0) = |\vec{v}|^2$ and similarly $\vec{w} \cdot \vec{w} = |\vec{w}|^2$, we get that

$$\begin{aligned}
(\vec{v} + \vec{w}) \cdot (\vec{v} + 2\vec{w}) &= (\vec{v} + \vec{w}) \cdot (\vec{v}) + (\vec{v} + \vec{w}) \cdot (2\vec{w}) \\
&= (\vec{v} \cdot \vec{v} + \vec{w} \cdot \vec{v}) + 2(\vec{v} \cdot \vec{w} + \vec{w} \cdot \vec{w}) \\
&= (|\vec{v}|^2 + \vec{w} \cdot \vec{v}) + 2(\vec{v} \cdot \vec{w} + |\vec{w}|^2) \\
&= |\vec{v}|^2 + 2|\vec{w}|^2 + 3(\vec{v} \cdot \vec{w}) \\
&= (3)^2 + 2(5)^2 + 3(10) \\
&= \boxed{89}
\end{aligned}$$

88. Find parametric equations for the line that is the intersection of the two planes $x + y = 3$ and $y + z = 1$.

The point $(2, 1, 0)$ is on both planes, so it is part of their intersection. The line will be perpendicular to the normal vectors for each plane. That is, it will be perpendicular to $\langle 1, 1, 0 \rangle$ and $\langle 0, 1, 1 \rangle$.

"The line through $(2, 1, 0)$ perpendicular to $\langle 1, 1, 0 \rangle$ and $\langle 0, 1, 1 \rangle$" is exactly Problem #84. Answer: $\boxed{x = 2 + t, \quad y = 1 - t, \quad z = t}$.