Linear Algebra, Winter 2021

List 6

Lines and planes, cross product

The vector $\vec{r}$ is used for $\langle x, y\rangle$ in 2D and $\langle x, y, z\rangle$ in 3D.

The line through point $\vec{p}$ parallel to vector $\vec{d}$ has parametric vector equation

$$\vec{r} = \vec{p} + t\vec{d}.$$

The vector $\vec{d}$ is called a **direction vector** for the line. This vector equation is a short way of writing the 2 or 3 parametric equations with only scalars.

The plane through point $\vec{p}$ parallel to vectors $\vec{u}$ and $\vec{v}$ has parametric vector eqn.

$$\vec{r} = \vec{p} + t\vec{u} + s\vec{v}.$$

The plane through $\vec{p}$ perpendicular to vector $\vec{n}$ has vector equation

$$\vec{n} \cdot (\vec{r} - \vec{p}) = 0.$$

The vector $\vec{n}$ is called a **normal vector** for the plane. If $\vec{p} = \langle x_0, y_0, z_0\rangle$ and $\vec{n} = \langle a, b, c\rangle$, the vector equation above is equivalent to the "scalar equation"

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

or to

$$ax + by + cz = D$$

for the number $D = \vec{p} \cdot \vec{n}$.

71. Write an equation in vector form for the line through points $(2, 3)$ and $(-3, 7)$.

72. Describe the line through the point $(2,\ 2.4,\ 3.5)$ and parallel to the vector $3\,\vec{\imath} + 2\,\vec{\jmath} - \vec{k}$ using the parameter t in
    (a) one vector equation.
    (b) three scalar equations.

73. Does the point $(4, 8, 7)$ line on the line $\vec{r} = \langle 1, 2, 6\rangle + \langle 3, 8, 9\rangle t$?

74. Find a scalar equation for the plane through the origin perpendicular to the vector $\langle 1, -2, 5\rangle$.

75. Find an equation for the plane through the point $(1, -1, -1)$ parallel to the plane $5x - y - z = 6$.

76. Find the distance from the point $(-6, 3, 5)$ to the plane $x - 2y - 4z = 8$.

77. Explain why the parametric equations

$$\begin{cases} x = 1 - t \\ y = 2 - 3t \\ z = 4t \end{cases} \quad \text{and} \quad \begin{cases} x = 2s \\ y = -1 + 6s \\ z = 4 - 8s \end{cases}$$

describe the same line.

78. Find the intersection point of the line

$$L: \qquad x = t,\ y = 1 - 2t,\ z = -3 + 2t$$

and the plane

$$P: \qquad 3x - y - 2z = 3.$$

79. Do the lines

$$\begin{aligned} L_1: &\quad x = 3 + t,\ y = 2 - 2t,\ z = 1 + 5t \\ L_2: &\quad x = 8 - 3s,\ y = 1 + s,\ z = -10 + 5s \end{aligned}$$

intersect? If so, find the intersection point and the angle between the two lines at that point (using a calculator for $\cos^{-1}$).

80. Do the lines

$$\begin{aligned} L_1: &\quad x = 3 + t,\ y = 2 - 2t,\ z = 1 + 5t \\ L_3: &\quad x = 5 + 2s,\ y = -6 - s,\ z = 7 - 4s \end{aligned}$$

intersect? If so, find the intersection point and the angle between the two lines at that point (using a calculator for $\cos^{-1}$).

81. Give parametric equations (with two parameters) for the plane through the origin parallel to both $\langle 1, 5, 9 \rangle$ and $\langle 2, -1, 2 \rangle$.

The **cross product** of $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ is written $\vec{a} \times \vec{b}$ and can be calculated as

$$\vec{a} \times \vec{b} = (a_2 b_3 - a_3 b_2)\vec{\imath} + (a_3 b_1 - a_1 b_3)\vec{\jmath} + (a_1 b_2 - a_2 b_1)\vec{k}.$$

This vector is perpendicular to $\vec{a}$ and to $\vec{b}$, has length $|\vec{a}||\vec{b}|\sin(\theta)$ where θ is the angle between $\vec{a}$ and $\vec{b}$, and its direction obeys the Right-Hand Rule.

82. For each expression below, does it represent a scalar, a vector, or nonsense?

(a) $\vec{a} \times t$

(b) $\vec{a} \times \vec{b}$

(c) $(\vec{a} \times b) \times \vec{c}$

(d) $(\vec{u} + \vec{w}) \times \vec{v}$

(e) $(\vec{u} \times \vec{w}) + \vec{v}$

(f) $t(\vec{u} \times \vec{w}) + s\vec{v}$

(g) $a\vec{u} \times (b\vec{v} + c\vec{w})$

(h) $\begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix} \times \vec{w}$

(i) $\langle 1, 3, 8 \rangle \times (\vec{a} + \vec{b})$

(j) $\langle t, 4, s \rangle \times (\vec{a} - \vec{b})$

(k) $|\vec{v} \times \vec{w}|$

(l) $5\vec{a} + \left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \times \vec{a} \right)$

(m) $||\vec{v} \times \vec{w}||$

(n) $\vec{0} \times \vec{0}$

(o) $(\vec{u} \times \vec{v}) \cdot \vec{w}$

(p) $\vec{u} \times (\vec{v} \cdot \vec{w})$

(q) $|\langle 7, 4, 2 \rangle| \times \vec{v}$

(r) $\vec{a} \times \vec{b} + t$

(s) $|\vec{v} \times \vec{w}| - |\vec{v} \cdot \vec{w}|$

83. For the vectors $\vec{v} = \langle 2, -1, 3 \rangle$ and $\vec{w} = \langle 4, 2, 1 \rangle$,

(a) Give the vector $\vec{v} \times \vec{w}$.

(b) Give the vector $\vec{w} \times \vec{v}$.

(c) Give the scalar $\vec{v} \cdot (\vec{v} \times \vec{w})$.

(d) Give the scalar $\vec{w} \cdot (\vec{v} \times \vec{w})$.

84. Find parametric equations for the line through $(2, 1, 0)$ perpendicular to both $\vec{\imath}+\vec{\jmath}$ and $\vec{\jmath}+\vec{k}$.

85. Give a scalar equation (with no parameters) for the plane from #81.

86. If $\vec{a} \cdot \vec{b} = \sqrt{3}$ and $\vec{a} \times \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, find the angle between $\vec{a}$ and $\vec{b}$.

87. If $|\vec{v}| = 3$, $|\vec{w}| = 5$, and $\vec{v} \cdot \vec{w} = 10$, compute $(\vec{v} + \vec{w}) \cdot (\vec{v} + 2\vec{w})$.

88. Find parametric equations for the line that is the intersection of the two planes $x + y = 3$ and $y + z = 1$.