Linear Algebra, Winter 2021

List 8

Determinant, inverse, systems of equations

101. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$, $C = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $D = \begin{bmatrix} 0 & 5 & 2 \end{bmatrix}$, and $E = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 1 \end{bmatrix}$.
Write all the products of two matrices from this list that exist (e.g., AA exists, but AC does not).

    There are 9 valid products of this form: $\boxed{AA, AB, BA, BB, CD, DC, DE, EA, EB}$.

102. If $\begin{bmatrix} 3 & 5 \\ 5 & 9 \end{bmatrix} M = \begin{bmatrix} 8 & 25 & 12 \\ 14 & 45 & 22 \end{bmatrix}$, what are the dimensions of matrix M? $\boxed{2 \times 3}$

103. Give the dimensions of the matrix $\begin{bmatrix} 2 & -8 \\ 1 & 5 \\ 0 & -7 \end{bmatrix} \begin{bmatrix} 9 & 0 & 0 & 11 & 4 \\ -2 & -8 & 6 & 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 5 \\ 4 \\ 0 \\ 1 \\ -9 \end{bmatrix} \begin{bmatrix} \frac{2}{7} & -1 & \frac{4}{7} \end{bmatrix}$.

    (Do *not* compute the matrix product.)
    Dimensions $(3 \times 2)(2 \times 5)(5 \times 1)(1 \times 3)$ leads to $\boxed{3 \times 3}$.

104. For each of the following equations, either give the dimensions of the matrix M or state that such a matrix does not exist.

    (a) $M = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ $\boxed{2 \times 1}$

    (b) $M = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$ $\boxed{\text{doesn't exist}}$

    (c) $M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ $\boxed{\text{doesn't exist}}$

    (d) $M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ $\boxed{3 \times 2}$

    (e) $M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\boxed{2 \times 1}$

    (f) $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} M \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\boxed{1 \times 3}$

(g) $\begin{bmatrix}1 & 2 & 3\end{bmatrix} M \begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$ doesn't exist

105. Find the cosine of the angle between $\begin{bmatrix}3\\ 1\end{bmatrix}$ and $\begin{bmatrix}4 & -5\\ 1 & -3\end{bmatrix}\begin{bmatrix}3\\ 1\end{bmatrix}$.

106. (a) Is $\begin{bmatrix}2 & 5\\ 9 & -2\end{bmatrix}\begin{bmatrix}4\\ 4\end{bmatrix}$ parallel to $\begin{bmatrix}4\\ 4\end{bmatrix}$? Yes

(b) Is $\begin{bmatrix}2 & 5\\ 9 & -2\end{bmatrix}\begin{bmatrix}2\\ 5\end{bmatrix}$ parallel to $\begin{bmatrix}2\\ 5\end{bmatrix}$? No

(c) Is $\begin{bmatrix}2 & 5\\ 9 & -2\end{bmatrix}\begin{bmatrix}2\\ 9\end{bmatrix}$ parallel to $\begin{bmatrix}2\\ 9\end{bmatrix}$? No

(d) Is $\begin{bmatrix}2 & 5\\ 9 & -2\end{bmatrix}\begin{bmatrix}5\\ -9\end{bmatrix}$ parallel to $\begin{bmatrix}5\\ -9\end{bmatrix}$? Yes

(e) Is $\begin{bmatrix}2 & 5\\ 9 & -2\end{bmatrix}\begin{bmatrix}2\\ -9\end{bmatrix}$ parallel to $\begin{bmatrix}2\\ -9\end{bmatrix}$? No

(f) Is $\begin{bmatrix}2 & 5\\ 9 & -2\end{bmatrix}\begin{bmatrix}2\\ -2\end{bmatrix}$ parallel to $\begin{bmatrix}2\\ -2\end{bmatrix}$? No

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107. Compute the determinants of the following matrices.

(a) $\det\left(\begin{bmatrix}3 & 1\\ 2 & 5\end{bmatrix}\right) = 13$ (b) $\det\left(\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\right) = -2$ (c) $\det\left(\begin{bmatrix}3 & a\\ 2 & 5\end{bmatrix}\right) = 15 - 2a$

(d) $\det\left(\begin{bmatrix}1 & 3 & 5\\ 2 & 4 & 9\\ 5 & 6 & 8\end{bmatrix}\right) = 25$ (You may use a calculator for this.)

(e) $\det\left(\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}\right) = (\cos\alpha)^2 + (\sin\alpha)^2 = 1$

108. Determinants have several nice properties, such as $\det(AB) = \det(A)\cdot\det(B)$ and $\det(sA) = s^n A$ when A is an $n\times n$ matrix.

Suppose M is a 5×5 matrix with $\det(M) = 2$.

(a) Compute $\det(2M)$. $2^5\cdot 2 = 64$

(b) Compute $\det(-3M^2)$. $(-3)^5\cdot 2\cdot 2 = -972$

(c) Compute $\det(M^{-1})$. $\frac{1}{2}$

109. Which of the following matrices have an inverse?

(A) Matrix A, a 3×3 matrix with $\det(A) = 3$.

(B) Matrix B, a 3×5 matrix where every number in the matrix is 1.

(C) Matrix C, a 4×4 matrix where every number in the matrix is 0.

(D) Matrix D, a 5×5 matrix with $\det(D) = -1$.

(E) Matrix E, a 7×7 matrix with $\det(E) = 0$.

only A and D have an inverse

110. For what values of the parameter p are the following matrices invertible? Give a formula for the inverse of each matrix.

(a) $\begin{bmatrix} 1 & 2 \\ p & p^3 \end{bmatrix}$ If $p \neq 0, \sqrt{2}, -\sqrt{2}$, inverse is $\frac{1}{p^3-2p} \begin{bmatrix} p^3 & -2 \\ -p & 1 \end{bmatrix}$.

(b) $\begin{bmatrix} \cos p & -\sin p \\ \sin p & \cos p \end{bmatrix}$ Inverse is $\begin{bmatrix} \cos p & \sin p \\ -\sin p & \cos p \end{bmatrix}$.

(c) $\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} - pI_{2\times 2}$ If $p \neq 4, -1$, inverse is $\frac{1}{p^2-3p-4} \begin{bmatrix} 2-p & -2 \\ -3 & 1-p \end{bmatrix}$.

111. For $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$, we have $A^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}^{-1} = \begin{bmatrix} -4 & 3 \\ 7/2 & -5/2 \end{bmatrix}$.

(a) Calculate AB.

(b) Calculate $(AB)^{-1}$, that is, the inverse of the matrix from part (a).

(c) Calculate $A^{-1}B^{-1}$.

(d) Calculate $B^{-1}A^{-1}$.

(e) Is $(AB)^{-1} = A^{-1}B^{-1}$ true? Is $(AB)^{-1} = B^{-1}A^{-1}$ true?

112. Solve the system of equations $\begin{cases} 4x + y = 25 \\ x - 6y = 25. \end{cases}$ $x = 7, y = -3$

113. Write the system of three equations that corresponds to the matrix equation

$$\begin{bmatrix} 15 & -2 & 46 \\ 0 & 14 & 15 \\ 13 & 7 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 19 \\ -4 \end{bmatrix}.$$

Do *not* try to solve the system. (This was originally #101.)

$$\begin{cases} 15x - 2y + 46z = 0 \\ 14y + 15z = 19 \\ 13x + 7y - z = -4 \end{cases}$$

114. Given that

$$\begin{bmatrix} 6 & 1 & 5 \\ 0 & 2 & 9 \\ -1 & 4 & 18 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 2 & -1 \\ -9 & 113 & -54 \\ 2 & -25 & 12 \end{bmatrix},$$

solve the following systems of equations *without a calculator*:

(a) $\begin{cases} 6x + y + 5z = -5 \\ 2y + 9z = 0 \\ -x + 4y + 18z = 1. \end{cases}$ $(x, y, z) = \begin{bmatrix} 0 & 2 & -1 \\ -9 & 113 & -54 \\ 2 & -25 & 12 \end{bmatrix} \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix} = (-1, -9, 2)$

(b) $\begin{cases} 6x + y + 5z = 1 \\ 2y + 9z = 1 \\ -x + 4y + 18z = 7. \end{cases}$ $(x, y, z) = \begin{bmatrix} 0 & 2 & -1 \\ -9 & 113 & -54 \\ 2 & -25 & 12 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 7 \end{bmatrix} = (-5, -274, 61)$

(c) $\begin{cases} 6x + y + 5z = -3 \\ 2y + 9z = -3 \\ -x + 4y + 18z = 9. \end{cases}$ $(x, y, z) = (-15, -798, 177)$

(d) $\begin{cases} 6x + y + 5z = 1 \\ 2y + 9z = 1 \\ -x + 4y + 18z = -10. \end{cases}$ $(x, y, z) = (12, 644, -143)$

115. A collection of 1 zł and 2 zł coins totals 38 zł, and there are 20 coins all together. How many of each coin are there?

System: $\begin{cases} 1j + 2d = 38 \\ j + d = 20. \end{cases}$ Solution: $(j, d) = (2, 18)$.
Two 1-zł-coins and twenty 2-zł-coints.

116. A band is selling tickets for a concert at two prices, normal and reduced. On Tuesday, they sell 31 normal tickets and 24 reduced tickets for a total of 2820 zł. On Wednesday, they sell 28 normal and 27 reduced for 2760 zł total. What are the prices for individual tickets?

System: $\begin{cases} 31n + 24r = 2820 \\ 28n + 27r = 2760. \end{cases}$ Solution: $(n, r) = (60, 40)$.
Normal tickets are 60 zł and reduced tickets are 40 zł.

117. Sam has a secret two-digit number and gives the following clues: the sum of the digits is 9, and if you reverse the digits you get a number that is 45 more than the original number. Find Sam's secret number.

A number written "ab" has the value $10a + b$. Its reverse "ba" has value $10b + a$.
System: $\begin{cases} a + b = 9 \\ (10b + a) - (10a + b) = 45. \end{cases} \rightarrow \begin{cases} a + b = 9 \\ -9a + 9b = 45. \end{cases}$
Solution: $(a, b) = (2, 7)$, so the secret number is 27.

118. A menu has the following options:

1 coffee 2 muffins 1 apple	2 coffees 5 muffins	2 coffees 2 muffins 2 apples
24.00 zł	47.25 zł	37.50 zł

Assuming there are no deals/discounts, how much does each item cost?

System: $\begin{cases} c + 2m + a = 24 \\ 2c + 5m = 47.25 \\ 2c + 2m + 2a = 37.50. \end{cases}$ Solution: $\begin{array}{r} c = 3 \\ m = 10.5 \\ a = 5.25 \end{array}$
A coffee costs 3.00 zł, a muffin costs 10.50 zł, and an apple costs 5.25 zł.