Linear Algebra, Winter 2021

List 8

Determinant, inverse, systems of equations

101. Let $A = \begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$, $B = \begin{bmatrix}2 & 3\\ 0 & 5\end{bmatrix}$, $C = \begin{bmatrix}1\\ 2\\ 1\end{bmatrix}$, $D = \begin{bmatrix}0 & 5 & 2\end{bmatrix}$, and $E = \begin{bmatrix}1 & 0\\ 0 & 2\\ 3 & 1\end{bmatrix}$. Write all the products of two matrices from this list that exist (e.g., AA exists, but AC does not).

102. If $\begin{bmatrix}3 & 5\\ 5 & 9\end{bmatrix} M = \begin{bmatrix}8 & 25 & 12\\ 14 & 45 & 22\end{bmatrix}$, what are the dimensions of matrix M?

103. Give the dimensions of the matrix $\begin{bmatrix}2 & -8\\ 1 & 5\\ 0 & -7\end{bmatrix} \begin{bmatrix}9 & 0 & 0 & 11 & 4\\ -2 & -8 & 6 & 1 & \frac{1}{2}\end{bmatrix} \begin{bmatrix}5\\ 4\\ 0\\ 1\\ -9\end{bmatrix} \begin{bmatrix}\frac{2}{7} & -1 & \frac{4}{7}\end{bmatrix}$.

(Do *not* compute the matrix product.)

104. For each of the following equations, either give the dimensions of the matrix M or state that such a matrix does not exist.

(a) $M = \begin{bmatrix}1 & 2 & 3 & 4\\ 5 & 6 & 7 & 8\end{bmatrix} \begin{bmatrix}1\\ 2\\ 3\\ 4\end{bmatrix}$

(b) $M = \begin{bmatrix}1\\ 2\\ 3\\ 4\end{bmatrix} \begin{bmatrix}1 & 2 & 3 & 4\\ 5 & 6 & 7 & 8\end{bmatrix}$

(c) $M = \begin{bmatrix}1 & 2\\ 3 & 4\\ 5 & 6\end{bmatrix} \begin{bmatrix}1 & 2\\ 3 & 4\\ 5 & 6\end{bmatrix}$

(d) $M = \begin{bmatrix}1 & 2\\ 3 & 4\\ 5 & 6\end{bmatrix} + \begin{bmatrix}1 & 2\\ 3 & 4\\ 5 & 6\end{bmatrix}$

(e) $M = \begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix} \begin{bmatrix}1\\ 2\end{bmatrix} + \begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix} \begin{bmatrix}1\\ 2\\ 3\end{bmatrix}$

(f) $\begin{bmatrix}1\\ 2\\ 3\\ 4\\ 5\end{bmatrix} M \begin{bmatrix}1 & 2\\ 3 & 4\\ 5 & 6\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\\ 0 & 0\\ 0 & 0\\ 0 & 0\end{bmatrix}$

(g) $\begin{bmatrix}1 & 2 & 3\end{bmatrix} M \begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$

105. Find the cosine of the angle between $\begin{bmatrix}3\\ 1\end{bmatrix}$ and $\begin{bmatrix}4 & -5\\ 1 & -3\end{bmatrix} \begin{bmatrix}3\\ 1\end{bmatrix}$.

106. (a) Is $\begin{bmatrix}2 & 5\\ 9 & -2\end{bmatrix} \begin{bmatrix}4\\ 4\end{bmatrix}$ parallel to $\begin{bmatrix}4\\ 4\end{bmatrix}$?

(b) Is $\begin{bmatrix}2 & 5\\ 9 & -2\end{bmatrix} \begin{bmatrix}2\\ 5\end{bmatrix}$ parallel to $\begin{bmatrix}2\\ 5\end{bmatrix}$?

(c) Is $\begin{bmatrix}2 & 5\\ 9 & -2\end{bmatrix} \begin{bmatrix}2\\ 9\end{bmatrix}$ parallel to $\begin{bmatrix}2\\ 9\end{bmatrix}$?

(d) Is $\begin{bmatrix}2 & 5\\ 9 & -2\end{bmatrix} \begin{bmatrix}5\\ -9\end{bmatrix}$ parallel to $\begin{bmatrix}5\\ -9\end{bmatrix}$?

(e) Is $\begin{bmatrix}2 & 5\\ 9 & -2\end{bmatrix} \begin{bmatrix}2\\ -9\end{bmatrix}$ parallel to $\begin{bmatrix}2\\ -9\end{bmatrix}$?

(f) Is $\begin{bmatrix}2 & 5\\ 9 & -2\end{bmatrix} \begin{bmatrix}2\\ -2\end{bmatrix}$ parallel to $\begin{bmatrix}2\\ -2\end{bmatrix}$?

107. Compute the determinants of the following matrices.

(a) $\begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ (c) $\begin{bmatrix} 3 & a \\ 2 & 5 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 9 \\ 5 & 6 & 8 \end{bmatrix}$ (You may use a calculator for this.) (e) $\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

108. Determinants have several nice properties, such as $\det(AB) = \det(A) \cdot \det(B)$ and $\det(sA) = s^n A$ when A is an $n \times n$ matrix.

Suppose M is a 5×5 matrix with $\det(M) = 2$.

(a) Compute $\det(2M)$.

(b) Compute $\det(-3M^2)$.

(c) Compute $\det(M^{-1})$.

109. Which of the following matrices have an inverse?

(A) Matrix A, a 3×3 matrix with $\det(A) = 3$.

(B) Matrix B, a 3×5 matrix where every number in the matrix is 1.

(C) Matrix C, a 4×4 matrix where every number in the matrix is 0.

(D) Matrix D, a 5×5 matrix with $\det(D) = -1$.

(E) Matrix E, a 7×7 matrix with $\det(E) = 0$.

110. For what values of the parameter p are the following matrices invertible? Give a formula for the inverse of each matrix.

(a) $\begin{bmatrix} 1 & 2 \\ p & p^3 \end{bmatrix}$ (b) $\begin{bmatrix} \cos p & -\sin p \\ \sin p & \cos p \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} - pI_{2\times 2}$

111. For $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$, we have $A^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}^{-1} = \begin{bmatrix} -4 & 3 \\ 7/2 & -5/2 \end{bmatrix}$.

(a) Calculate AB.

(b) Calculate $(AB)^{-1}$, that is, the inverse of the matrix from part (a).

(c) Calculate $A^{-1}B^{-1}$.

(d) Calculate $B^{-1}A^{-1}$.

(e) Is $(AB)^{-1} = A^{-1}B^{-1}$ true? Is $(AB)^{-1} = B^{-1}A^{-1}$ true?

112. Solve the system of equations $\begin{cases} 4x + y = 25 \\ x - 6y = 25. \end{cases}$

113. Write the system of three equations that corresponds to the matrix equation

$$\begin{bmatrix} 15 & -2 & 46 \\ 0 & 14 & 15 \\ 13 & 7 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 19 \\ -4 \end{bmatrix}.$$

Do *not* try to solve the system. (This was originally #101.)

114. Given that

$$\begin{bmatrix} 6 & 1 & 5 \\ 0 & 2 & 9 \\ -1 & 4 & 18 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 2 & -1 \\ -9 & 113 & -54 \\ 2 & -25 & 12 \end{bmatrix},$$

solve the following systems of equations *without a calculator*:

(a) $\begin{cases} 6x + y + 5z = -5 \\ 2y + 9z = 0 \\ -x + 4y + 18z = 1. \end{cases}$

(b) $\begin{cases} 6x + y + 5z = 1 \\ 2y + 9z = 1 \\ -x + 4y + 18z = 7. \end{cases}$

(c) $\begin{cases} 6x + y + 5z = -3 \\ 2y + 9z = -3 \\ -x + 4y + 18z = 9. \end{cases}$

(d) $\begin{cases} 6x + y + 5z = 1 \\ 2y + 9z = 1 \\ -x + 4y + 18z = -10. \end{cases}$

115. A collection of 1 zł and 2 zł coins totals 38 zł, and there are 20 coins all together. How many of each coin are there?

116. A band is selling tickets for a concert at two prices, normal and reduced. On Tuesday, they sell 31 normal tickets and 24 reduced tickets for a total of 2820 zł. On Wednesday, they sell 28 normal and 27 reduced for 2760 zł total. What are the prices for individual tickets?

117. Sam has a secret two-digit number and gives the following clues: the sum of the digits is 9, and if you reverse the digits you get a number that is 45 more than the original number. Find Sam's secret number.

118. A boat has a motor that would make it move at constant speed in still water. One day, the boat is sailed 280 km downstream[1] and back; the trip downstream took 4 hours, and the trip upstream took 7 hours. What is the speed of the boat in still water, and what is the speed of the river's current?

119. A menu has the following options:

1 coffee 2 muffins 1 apple	2 coffees 5 muffins	2 coffees 2 muffins 2 apples
24.00 zł	47.25 zł	37.50 zł

Assuming there are no deals/discounts, how much does each item cost?

[1] When traveling downstream, the boat's speed is its still-water speed plus the speed of the river current. When traveling upstream, the boat's speed is its still-water speed minus the river current.