Linear Algebra, Winter 2021

# List 9

*Linear independence, rank, eigenvalues and eigenvectors*

A collection of vectors is **linearly dependent** if one vector in the collection is a linear combination of the others (that is, a sum of scalar multiples of those vectors); otherwise it is **linearly independent**. A collection of one non-zero vector is always linearly independent.

120. (a) Is the collection $\{[-1,8,8]\}$ linearly independent? Yes
    (b) Is the collection $\{[-1,8,8],[5,0,0]\}$ linearly independent? Yes
    (c) Is the collection $\{[-1,8,8],[5,0,0],[3,1,3]\}$ linearly independent? Yes
    (d) Is the collection $\{[-1,8,8],[5,0,0],[3,1,3],[3,-4,4]\}$ linearly independent? No

121. (a) Is the collection $\{[0,2,5]\}$ linearly independent? Yes
    (b) Is the collection $\{[0,2,5],[1,1,-4]\}$ linearly independent? Yes
    (c) Is the collection $\{[0,2,5],[1,1,-4],[2,4,-3]\}$ linearly independent? No
    (d) Is the collection $\{[0,2,5],[1,1,-4],[2,4,-3],[2,8,7]\}$ linearly independent? No

The **rank** of a matrix is the maximum number of linearly independent rows in the matrix. This is also the maximum number of linearly independent columns.

122. Give the rank of the following matrices: (a) 1, (b) 2, (c) 3, (d) 3

(a) $\begin{bmatrix}-1 & 8 & 8\end{bmatrix}$ (b) $\begin{bmatrix}-1 & 8 & 8\\ 5 & 0 & 0\end{bmatrix}$ (c) $\begin{bmatrix}-1 & 8 & 8\\ 5 & 0 & 0\\ 3 & 1 & 3\end{bmatrix}$ (d) $\begin{bmatrix}-1 & 8 & 8\\ 5 & 0 & 0\\ 3 & 1 & 3\\ 3 & -4 & 4\end{bmatrix}$

123. Give the rank of the following matrices: (a) 1, (b) 2, (c) 2, (d) 2

(a) $\begin{bmatrix}0 & 2 & 5\end{bmatrix}$ (b) $\begin{bmatrix}0 & 2 & 5\\ 1 & 1 & -4\end{bmatrix}$ (c) $\begin{bmatrix}0 & 2 & 5\\ 1 & 1 & -4\\ 2 & 4 & -3\end{bmatrix}$ (d) $\begin{bmatrix}0 & 2 & 5\\ 1 & 1 & -4\\ 2 & 4 & -3\\ 2 & 8 & 7\end{bmatrix}$

124. Calculate $\operatorname{rank}\left(\begin{bmatrix}4 & 0 & 0\\ 1 & 3 & 5\\ 6 & 3 & 2\end{bmatrix}\right)$. The column $\begin{bmatrix}4\\1\\6\end{bmatrix}$ cannot be a linear combination of the other two because they both have a 0 in the first entry.
    The column $\begin{bmatrix}0\\5\\2\end{bmatrix}$ cannot be a linear com. of only $\begin{bmatrix}0\\3\\3\end{bmatrix}$ because all multiples of $\begin{bmatrix}0\\3\\3\end{bmatrix}$ have the same $y$- and $z$-component. Therefore all 3 columns are linearly independent, so the rank is 3.

125. For which values of $p$ does the matrix $\begin{bmatrix} 4 & 0 & 0 \\ 1 & 3 & 5 \\ 6 & 3 & p \end{bmatrix}$ have rank 2? $\boxed{p = 5}$ For which values does it have rank 3? $\boxed{\text{all } p \neq 5}$ Rank 1? $\boxed{\text{None}}$

126. Find rank$(A)$ and det$(A)$ for the matrix $A = \begin{bmatrix} 4 & 8 & 1 \\ 7 & 14 & 6 \\ 1 & 2 & 7 \end{bmatrix}$ without a calculator.

$\boxed{\text{rank}(A) = 2}$ because the second column is two times the first column.

$\boxed{\det(A) = 0}$ because the rank is less than the number of columns.

127. The rank of $\left[\begin{smallmatrix} 4 & 1 & 5 \\ 2 & 0 & 2 \\ 1 & -1 & 0 \end{smallmatrix}\right]$ is 2 and the rank of $\left[\begin{smallmatrix} 4 & 1 & 5 & 1 \\ 2 & 0 & 2 & 2 \\ 1 & -1 & 0 & 7 \end{smallmatrix}\right]$ is 3.

How many solutions does the system $\begin{cases} 4x + y + 5z = 1 \\ 2x + \phantom{y+{}} 2z = 2 \\ x - y \phantom{+5z} = 7 \end{cases}$ have? None

128. The rank of $\left[\begin{smallmatrix} 4 & 1 & 5 \\ 2 & 0 & 2 \\ 1 & -1 & 0 \end{smallmatrix}\right]$ is 2 and the rank of $\left[\begin{smallmatrix} 4 & 1 & 5 & 6 \\ 2 & 0 & 2 & 4 \\ 1 & -1 & 0 & 4 \end{smallmatrix}\right]$ is also 2.

How many solutions does the system $\begin{cases} 4x + y + 5z = 6 \\ 2x + \phantom{y+{}} 2z = 4 \\ x - y \phantom{+5z} = 4 \end{cases}$ have? Infinitely many

129. The rank of $\left[\begin{smallmatrix} 7 & 2 & 5 \\ 9 & 0 & 3 \\ 3 & -1 & 0 \end{smallmatrix}\right]$ is 3 and the rank of $\left[\begin{smallmatrix} 7 & 2 & 5 & -1 \\ 9 & 0 & 3 & 0 \\ 3 & -1 & 0 & 5 \end{smallmatrix}\right]$ is also 3.

How many solutions does the system $\begin{cases} 7x + 2y + 5z = -1 \\ 9x + \phantom{2y+{}} 3z = 0 \\ 3x - \phantom{2}y \phantom{+5z} = 5 \end{cases}$ have? One

130. If the numbers $a, b, c, d$ are such that $(x, y) = (9, 1)$ is a solution to $\begin{cases} ax + by = 2 \\ cx + dy = 3 \end{cases}$ but the system $\begin{cases} ax + by = 4 \\ cx + dy = 9 \end{cases}$ has no solutions, what is the rank of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$?

Call this number $r$. Since the system $\begin{cases} ax + by = 4 \\ cx + dy = 9 \end{cases}$ has no solutions, $r < 2$. The values $9, 1, 2, 3$ are not important—the fact that $\begin{cases} ax + by = 2 \\ cx + dy = 3 \end{cases}$ simply means that $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ is not $\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and thus $r > 0$. The only integer $0 < r < 2$ is $r = \boxed{1}$.

Let $M$ be a square matrix. If $M\vec{v} = \lambda\vec{v}$ with $\vec{v} \neq \vec{0}$ then the vector $\vec{v}$ is called an **eigenvector** of $M$ and the number $s$ is called an **eigenvalue** of $M$.
- The eigenvalues of $M$ are exactly the numbers $\lambda$ for which $\det(A - \lambda I) = 0$.
- The determinant of $M$ is exactly equal to the product of all its eigenvalues.

131. Find the eigenvalues of $\begin{bmatrix} 4 & 1 \\ -2 & 8 \end{bmatrix}$.

$\det\left(\left[\begin{smallmatrix} 4-\lambda & 1 \\ -2 & 8-\lambda \end{smallmatrix}\right]\right) = (4 - \lambda)(8 - \lambda) - (1)(-2) = \lambda^2 - 12\lambda + 34.$

The roots of $\lambda^2 - 12\lambda + 34$ are $\boxed{6 + \sqrt{2} \text{ and } 6 - \sqrt{2}}$.

132. Find the eigenvalues of $\begin{bmatrix} 2 & 1 \\ 7 & 8 \end{bmatrix}$, then find the eigenvector for each eigenvalue.

$\det\left(\begin{bmatrix} 2-\lambda & 1 \\ 7 & 8-\lambda \end{bmatrix}\right) = 0$ gives $\lambda^2 - 10\lambda + 9 = 0$, so the eigenvalues are $\boxed{9 \text{ and } 1}$.

For $\lambda = 9$, we want $\begin{bmatrix} 2 & 1 \\ 7 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 9\begin{bmatrix} x \\ y \end{bmatrix}$, so $\begin{bmatrix} 2x+y \\ 7x+8y \end{bmatrix} = \begin{bmatrix} 9x \\ 9y \end{bmatrix}$. The solutions to $\begin{cases} 2x+y=9x \\ 7x+8y=9y \end{cases}$ are $\boxed{\text{any multiple of } \begin{bmatrix} 1 \\ 7 \end{bmatrix}}$. For $\lambda = 1$ we get $\boxed{\text{any multiple of } \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$.

133. Find the complex eigenvalues of $\begin{bmatrix} 4 & 1 \\ -8 & 8 \end{bmatrix}$.

$\det\left(\begin{bmatrix} 4-\lambda & 1 \\ -8 & 8-\lambda \end{bmatrix}\right) = (4-\lambda)(8-\lambda) - (1)(-8) = \lambda^2 - 12\lambda + 40.$

The roots of $\lambda^2 - 12\lambda + 40$ are $\boxed{6+2i \text{ and } 6-2i}$.

134. Give the determinant and rank of a $3 \times 3$ matrix whose eigenvalues are

(a) 6, $-2$, and 1. Determinant = product of eigenvalues. $(6)(-2)(1) = \boxed{-12}$. Rank $\boxed{3}$.

(b) 4, 3, and 0. det $= (4)(3)(0) = 0$, rank $= 2$ because 4 and 3 are not zero.

(c) $1-4i$, $1+4i$, and 2. det $= (1-4i)(1+4i)(2) = 34$, rank $= 3$ because all three are not zero.

135. If a $4 \times 4$ matrix of real numbers has $1-2i$ and $3+i$ as two of its eigenvalues, find its determinant.

$\det(A - \lambda I_{4\times4})$ will be a polynomial with real coefficients, so for each of its roots $a+bi$ the complex conjugate $a-bi$ will also be a root. Thus $1+2i$ and $3-i$ are also eigenvalues of the matrix. $(1-2i)(1+2i)(3+i)(3-i) = (5)(10) = \boxed{50}$

136. If the matrix $M$ satisfies

$$M\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -6 \\ 6 \end{bmatrix} \quad \text{and} \quad M\begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 12 \\ 6 \end{bmatrix} \quad \text{and} \quad M\begin{bmatrix} 7 \\ 2 \end{bmatrix} = \begin{bmatrix} 21 \\ 6 \end{bmatrix},$$

find the eigenvalues of $M$.

The middle equation is irrelevant because $[12, 6]$ is not parallel to $[5, 1]$. The other two equations show that $\boxed{-6 \text{ and } 3}$ are eigenvalues because

$$\begin{bmatrix} -6 \\ 6 \end{bmatrix} = (-6)\begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 21 \\ 6 \end{bmatrix} = 3\begin{bmatrix} 7 \\ 2 \end{bmatrix}.$$

137. If a $2 \times 2$ matrix $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ of real numbers satisfies

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 5 \\ 1+2i \end{bmatrix} = \begin{bmatrix} 20+10i \\ 10i \end{bmatrix},$$

find the (complex) eigenvalues of the matrix and find its determinant.

Eigenvalues $\boxed{4 \pm 2i}$ have product $\det(\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]) = (4+2i)(4-2i) = \boxed{20}$.