Linear Algebra, Winter 2021

List 9

Linear independence, rank, eigenvalues and eigenvectors

A collection of vectors is **linearly dependent** if one vector in the collection is a linear combination of the others (that is, a sum of scalar multiples of those vectors); otherwise it is **linearly independent**. A collection of one non-zero vector is always linearly independent.

120. (a) Is the collection $\{[-1, 8, 8]\}$ linearly independent?
   (b) Is the collection $\{[-1, 8, 8], [5, 0, 0]\}$ linearly independent?
   (c) Is the collection $\{[-1, 8, 8], [5, 0, 0], [3, 1, 3]\}$ linearly independent?
   (d) Is the collection $\{[-1, 8, 8], [5, 0, 0], [3, 1, 3], [3, -4, 4]\}$ linearly independent?
121. (a) Is the collection $\{[0, 2, 5]\}$ linearly independent?
   (b) Is the collection $\{[0, 2, 5], [1, 1, -4]\}$ linearly independent?
   (c) Is the collection $\{[0, 2, 5], [1, 1, -4], [2, 4, -3]\}$ linearly independent?
   (d) Is the collection $\{[0, 2, 5], [1, 1, -4], [2, 4, -3], [2, 8, 7]\}$ linearly independent?

The **rank** of a matrix is the maximum number of linearly independent rows in the matrix. This is also the maximum number of linearly independent columns.

122. Give the rank of the following matrices:

(a) $\begin{bmatrix} -1 & 8 & 8 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & 8 & 8 \\ 5 & 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} -1 & 8 & 8 \\ 5 & 0 & 0 \\ 3 & 1 & 3 \end{bmatrix}$ (d) $\begin{bmatrix} -1 & 8 & 8 \\ 5 & 0 & 0 \\ 3 & 1 & 3 \\ 3 & -4 & 4 \end{bmatrix}$

123. Give the rank of the following matrices:

(a) $\begin{bmatrix} 0 & 2 & 5 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 2 & 5 \\ 1 & 1 & -4 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 2 & 5 \\ 1 & 1 & -4 \\ 2 & 4 & -3 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 2 & 5 \\ 1 & 1 & -4 \\ 2 & 4 & -3 \\ 2 & 8 & 7 \end{bmatrix}$

124. Calculate $\text{rank}\left(\left[\begin{smallmatrix} 4 & 0 & 0 \\ 1 & 3 & 5 \\ 6 & 3 & 2 \end{smallmatrix}\right]\right)$.

125. For which values of p does the matrix $\begin{bmatrix} 4 & 0 & 0 \\ 1 & 3 & 5 \\ 6 & 3 & p \end{bmatrix}$ have rank 2? For which values does it have rank 3? Rank 1?

126. Find rank(A) and det(A) for the matrix $A = \begin{bmatrix} 4 & 8 & 1 \\ 7 & 14 & 6 \\ 1 & 2 & 7 \end{bmatrix}$ without a calculator.

127. The rank of $\left[\begin{smallmatrix} 4 & 1 & 5 \\ 2 & 0 & 2 \\ 1 & -1 & 0 \end{smallmatrix}\right]$ is 2 and the rank of $\left[\begin{smallmatrix} 4 & 1 & 5 & 1 \\ 2 & 0 & 2 & 2 \\ 1 & -1 & 0 & 7 \end{smallmatrix}\right]$ is 3.

How many solutions does the system $\begin{cases} 4x + y + 5z = 1 \\ 2x + \quad 2z = 2 \\ x - y \qquad = 7 \end{cases}$ have?

128. The rank of $\left[\begin{smallmatrix} 4 & 1 & 5 \\ 2 & 0 & 2 \\ 1 & -1 & 0 \end{smallmatrix}\right]$ is 2 and the rank of $\left[\begin{smallmatrix} 4 & 1 & 5 & 6 \\ 2 & 0 & 2 & 4 \\ 1 & -1 & 0 & 4 \end{smallmatrix}\right]$ is also 2.

How many solutions does the system $\begin{cases} 4x + y + 5z = 6 \\ 2x + \quad 2z = 4 \\ x - y \quad = 4 \end{cases}$ have?

129. The rank of $\left[\begin{smallmatrix} 7 & 2 & 5 \\ 9 & 0 & 3 \\ 3 & -1 & 0 \end{smallmatrix}\right]$ is 3 and the rank of $\left[\begin{smallmatrix} 7 & 2 & 5 & -1 \\ 9 & 0 & 3 & 0 \\ 3 & -1 & 0 & 5 \end{smallmatrix}\right]$ is also 3.

How many solutions does the system $\begin{cases} 7x + 2y + 5z = -1 \\ 9x + \quad 3z = 0 \\ 3x - \; y \quad = 5 \end{cases}$ have?

130. If the numbers a, b, c, d are such that $(x, y) = (9, 1)$ is a solution to $\begin{cases} ax + by = 2 \\ cx + dy = 3 \end{cases}$ but the system $\begin{cases} ax + by = 4 \\ cx + dy = 9 \end{cases}$ has no solutions, what is the rank of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$?

Let M be a square matrix. If $M\vec{v} = \lambda\vec{v}$ with $\vec{v} \neq \vec{0}$ then the vector $\vec{v}$ is called an **eigenvector** of M and the number s is called an **eigenvalue** of M. • The eigenvalues of M are exactly the numbers λ for which $\det(A - \lambda I) = 0$. • The determinant of M is exactly equal to the product of all its eigenvalues.

131. Find the eigenvalues of $\begin{bmatrix} 4 & 1 \\ -2 & 8 \end{bmatrix}$.

132. Find the eigenvalues of $\begin{bmatrix} 2 & 1 \\ 7 & 8 \end{bmatrix}$, then find the eigenvector for each eigenvalue.

133. Find the complex eigenvalues of $\begin{bmatrix} 4 & 1 \\ -8 & 8 \end{bmatrix}$.

134. Give the determinant and rank of a 3×3 matrix whose eigenvalues are

(a) 6, -2, and 1. (b) 4, 3, and 0. (c) $1 - 4i$, $1 + 4i$, and 2.

135. If a 4×4 matrix of real numbers has $1 - 2i$ and $3 + i$ as two of its eigenvalues, find its determinant.

136. If the matrix M satisfies

$$M\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -6 \\ 6 \end{bmatrix} \quad \text{and} \quad M\begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 12 \\ 6 \end{bmatrix} \quad \text{and} \quad M\begin{bmatrix} 7 \\ 2 \end{bmatrix} = \begin{bmatrix} 21 \\ 6 \end{bmatrix},$$

find the eigenvalues of M.

137. If a 2×2 matrix $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ of real numbers satisfies

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 5 \\ 1 + 2i \end{bmatrix} = \begin{bmatrix} 20 + 10i \\ 10i \end{bmatrix},$$

find the (complex) eigenvalues of the matrix and find its determinant.